Blessing of Dependence: Identifiability and Geometry of Discrete Models with Multiple Binary Latent Variables

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Abstract

Identifiability of discrete statistical models with latent variables is known to be challenging to study, yet crucial to a model’s interpretability and reliability. This work presents a general algebraic technique to investigate identifiability of complicated discrete models with latent and graphical components. Specifically, motivated by diagnostic tests collecting multivariate categorical data, we focus on discrete models with multiple binary latent variables. In the considered model, the latent variables can have arbitrary dependencies among themselves while the latent-to-observed measurement graph takes a “star-forest” shape. We establish necessary and sufficient graphical criteria for identifiability, and reveal an interesting and perhaps surprising phenomenon of blessing-of-dependence geometry: under the minimal conditions for generic identifiability, the parameters are identifiable if and only if the latent variables are not statistically independent. Thanks to this theory, we can perform formal hypothesis tests of identifiability in the boundary case by testing certain marginal independence of the observed variables. Our results give new understanding of statistical properties of graphical models with latent variables. They also entail useful implications for designing diagnostic tests or surveys that measure binary latent traits.

Keywords: Algebraic statistics, Contingency table, Diagnostic test, Generic identifiability, Graphical model, Hypothesis testing, Latent class model, Multivariate categorical data

1 Introduction

Discrete statistical models with latent variables and graphical components are widely used in statistics, machine learning, and many real-world applications. Examples include noisy-or Bayesian networks in medical diagnosis (Shwe et al., 1991; Halpern and Sontag, 2013), binary latent skill models in educational cognitive diagnosis (Chen et al., 2015; Xu, 2017; Gu and Xu, 2021a), and restricted Boltzmann machines and their variants in machine learning.
(Hinton et al., 2006; Goodfellow et al., 2016). Generally, incorporating latent variables into graphical models can greatly enhance the flexibility of a model. But such flexibility comes at a cost of the increasing model complexity and statistical subtlety, including the identifiability as a fundamental and challenging issue. Recognizing the potential non-identifiability caused by latent variables in graphical models, a body of works (e.g. Foygel et al., 2012; Evans and Richardson, 2014) project away the latent variables and study the property of the induced mixed graph among the observed variables. In some applications, however, the latent variables themselves carry important substantive meanings and inferring the parameters involving the latent is of paramount importance and practical interest (e.g. Bing et al., 2020a,b). In this work, we propose a general algebraic technique to investigate identifiability of discrete models with complicated latent and graphical components, characterize the minimal identifiability requirements for a class of such models motivated by diagnostic test applications, and along the way reveal a new geometry about multidimensional latent structures – the blessing of dependence on identifiability.

Statistically, a set of parameters for a family of models are said to be identifiable, if distinct values of the parameters correspond to distinct joint distributions of the observed variables. Identifiability is a fundamental prerequisite for valid statistical estimation and inference. In the literature, identifiability of discrete statistical models with latent variables is known to be challenging to study, partly due to their inherent complex nonlinearity. For example, Latent Class Models (LCMs; Lazarsfeld and Henry, 1968) are a simplest form of discrete models with latent structure, which assumes a univariate discrete latent variable renders the multivariate categorical responses conditional independent. Despite the seemingly simple structure and the popularity of LCMs in social and biomedical applications, their identifiability issues had eluded researchers for decades. Goodman (1974) investigated several specific small-dimensional LCMs, some being identifiable and some not. Gyllenberg et al. (1994) proved LCMs with binary responses are not strictly identifiable. Carreira-Perpinán and Renals (2000) empirically showed the so-called practical identifiability of LCMs using simulations. And finally, Allman et al. (2009) provided a rigorous statement about the generic identifiability of LCMs, whose proof leveraged Kruskal’s Theorem from Kruskal (1977) on the uniqueness of three-way tensor decompositions.

To be concrete, strict identifiability means model parameters are identifiable everywhere
in some parameter space $T$. A slightly weaker notion, \textit{generic identifiability} formalized and popularized by Allman et al. (2009), allows for a subset $\mathcal{N} \subseteq T$ where non-identifiability may occur, requiring $\mathcal{N}$ to have Lebesgue measure zero in $T$; such $\mathcal{N}$'s are basically zero sets of polynomials of model parameters (Allman et al., 2009). In some cases, these measure-zero subsets may be trivial, such as simply corresponding to the boundary of the parameter space. In some other cases, however, these subsets may be embedded in the interior of the parameter space, or even carries rather nontrivial geometry and interesting statistical interpretation (as is the case in this work under the minimal conditions for generic identifiability). A precise characterization of the measure-zero subset where identifiability breaks down is essential to performing correct statistical analysis and hypothesis testing (Drton, 2009). But it is often hard to obtain a complete understanding of such sets or to derive sharp conditions for identifiability in complicated latent variable models. These issues become even more challenging when graphical structures are present in latent variable models.

In this work, we present a general algebraic technique to study the identifiability of discrete statistical models with latent and graphical components, and use it to investigate an interesting class of such models. In the literature, pioneered by Allman et al. (2009), many existing identifiability results for models involving discrete latent structures leveraged Kruskal’s Theorem (e.g., Allman and Rhodes, 2008; Fang et al., 2019; Culpepper, 2019; Chen et al., 2020; Fang et al., 2020); these studies cover models ranging from phylogenetics in evolutionary biology to psychometrics in education and psychology. These identifiability proofs using Kruskal’s Theorem often rely on certain global rank conditions of the tensor formulated under the model. In contrast, we characterize a useful transformation property of the Khatri-Rao tensor products of arbitrary discrete variables’ probability tables. We then use this property to investigate how any specific parameter impacts the zero set of polynomials induced by the latent and graphical constraints. This general technique covers as a special case the one in Xu (2017) for restricted latent class models with binary responses. Our approach will unlock possibilities to study identifiability at the finest possible scale (rather than checking global rank conditions of tensors), and hence help obtain sharp conditions and characterize the aforementioned measure-zero non-identifiable sets. In particular, we will study settings where Kruskal’s theorem does not apply, demonstrating the power of this technique.
We provide an overview of our results. Motivated by epidemiological and educational diagnosis tests, we focus on discrete models with multiple binary latent variables, where the latent-to-observed measurement graph is a forest of star trees. Namely, each latent variable can have several observed noisy proxy variables as children. We allow the binary latent variables to have arbitrary dependencies among themselves for the greatest possible modeling flexibility. Call this model the Binary Latent cliquE Star foreSt (BLESS) model. We characterize the necessary and sufficient graphical criteria for strict and generic identifiability, respectively, of the BLESS model; this includes identifying both the discrete star-forest structure and the various continuous parameters. Under the minimal conditions for generic identifiability that each latent variable has exactly two observed children, we show that the measure-zero set \( \mathcal{N} \) in which identifiability breaks down is the independence model of the latent variables. That is, our identifiability condition delivers a deep and somewhat surprising geometry of blessing-of-dependence – the statistical dependence between latent variables can help restore identifiability. More broadly, this blessing-of-dependence phenomenon has nontrivial connections to and implications on the uniqueness of matrix and tensor decompositions. Building on the blessing of dependence, we propose a formal statistical hypothesis test of identifiability in the boundary case. In fact, in this case testing identifiability amounts to testing the marginal dependence of the latent variables’ observed children.

Our results have practical relevance on statistical modeling and real-world applications employing multidimensional latent structures. In many applications, it is intrinsically natural and interpretable to conceptualize each latent construct as presence or absence of some underlying trait. Examples include diagnosing the presence/absence of multiple unobserved disease pathogens of a patient in epidemiology (Wu et al., 2016, 2017; O’Brien et al., 2019), and determining the mastery/deficiency of multiple latent skills of a student in educational testing (von Davier, 2005; Henson et al., 2009; de la Torre, 2011; George and Robitzsch, 2015). Statistically, such an appealing conceptualization leads to statistical models with multidimensional binary latent variables. In addition, such models are also widely used in machine learning and deep learning as building blocks of deep generative models (Hinton and Salakhutdinov, 2006; Hinton et al., 2006; Salakhutdinov and Hinton, 2009). The models mentioned above often possess unique and curious algebraic structures. Understanding the statistical properties caused by these algebraic structures will provide valuable insight.
into scientific and statistical learning practices. This work contributes a new tool and new understanding in this regard.

The rest of this paper is organized as follows. Section 2 introduces the formal setup of the BLESS model and several relevant identifiability notions. Section 3 presents the main theoretical results of identifiability and overviews our general proof technique. Section 4 proposes a statistical hypothesis test of identifiability of the BLESS model under minimal conditions for generic identifiability. Section 5 presents two real-world examples and Section 6 concludes the paper.

2 Model Setup and Identifiability Notions

2.1 Binary Latent cliquE Star foreSt (BLESS) model

We next introduce the setup of the BLESS model, the focus of this study. We first introduce some notation. For an integer $m$, denote $[m] = \{1, \ldots, m\}$. For a $K$-dimensional vector $\mathbf{x} = (x_1, \ldots, x_K)$ and some index $k \in [K]$, denote the $(K-1)$-dimensional vector by $\mathbf{x}_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_K)$. Consider discrete statistical models with $K$ binary latent variables $\alpha_1, \ldots, \alpha_K \in \{0, 1\}$ and $p$ categorical observed variables $y_1, \ldots, y_p \in [d]$. Here $d \geq 2$ is the number of categories of each observed variable. Both the latent vector $\mathbf{\alpha} = (\alpha_1, \ldots, \alpha_K) \in \{0, 1\}^K$ and the observed vector $\mathbf{y} = (y_1, \ldots, y_p) \in [d]^K$ are subject-specific random quantities, and have their realizations for each subject $i$ in a random sample. For two random vectors (or variables) $\mathbf{x}$ and $\mathbf{y}$, denote by $\mathbf{x} \perp \perp \mathbf{y}$ if $\mathbf{x}$ and $\mathbf{y}$ are statistically independent, and denote by $\mathbf{x} \not\perp \not\perp \mathbf{y}$ otherwise.

A key structure in the BLESS model is the latent-to-observed measurement graph. This is a bipartite graph with directed edges from the latent $\alpha_k$’s to the observed $y_j$’s indicating direct statistical dependence. The BLESS model posits that the measurement graph is a forest of star trees; namely, each latent variable can have multiple observed proxy variables as children, but each observed variable has exactly one latent parent. On the latent part, we allow the binary latent variables to have arbitrary dependencies for the greatest possible modeling flexibility. Figure 1 provides a graphical illustration of the BLESS model, where we draw latent variables as white nodes and observed variables as gray nodes. Although
assuming each observed variable having exactly one latent parent may appear to be restrictive, we point out that the arbitrary dependence between the latent variables indeed allows the observables to have extremely flexible and rich joint distributions. In Figure 1, the solid directed edges from the latent to the observed variables form a star-forest-shaped measurement graph, and the dotted undirected edges between all pairs of latent variables indicate arbitrary possible dependence among them.

Equivalently, we can represent the bipartite measurement graph from the $K$ latent variables to the $p$ observed children in a $p \times K$ graphical matrix $G = (g_{j,k})$ with binary entries, where $g_{j,k} = 1$ indicates $\alpha_k$ is the latent parent of $y_j$ and $g_{j,k} = 0$ otherwise. Each row of $G$ contains exactly one entry of “1” due to the star-forest graph structure. Statistically, the conditional distribution of $y_j \mid \alpha$ equals that of $y_j \mid \alpha_k$ if and only if $g_{j,k} = 1$. We can therefore denote the conditional distribution of $y_j$ given the latent variables as follows,

$$\forall c_j \in [d], \quad P(y_j = c_j \mid \alpha, G) = P(y_j = c_j \mid \alpha_k, g_{j,k} = 1) = \begin{cases} \theta_{c_j|^1}, & \text{if } \alpha_k = 1; \\ \theta_{c_j|^0}, & \text{if } \alpha_k = 0. \end{cases}$$

To avoid the somewhat trivial non-identifiability issue associated with the sign flipping of
each binary latent variable ($\alpha_k$ flipping between 0 and 1), we assume

$$\theta_{c_j|1}^{(j)} > \theta_{c_j|0}^{(j)}, \quad c_j = 1, \ldots, d - 1,$$

for all $j \in [p]$; this could be understood as fixing the interpretation of $\alpha_k$ to that possessing the underlying latent trait always increases the response probability to the first $d - 1$ non-baseline categories. Fixing any other orders equally works for our identifiability arguments.

To complete the model specification, we need to describe the distribution of the latent variables $\alpha = (\alpha_1, \ldots, \alpha_K)$. As mentioned before, we do not impose any restrictions on the dependence structure among the latent variables, but just adopt the most flexible saturated model. That is, we give each possible binary latent pattern $\alpha \in \{0, 1\}^K$ a population proportion parameter $\nu_\alpha = \mathbb{P}(a_i = \alpha)$, where $a_i$ denoting the latent profile of a random subject $i$ in the population. The only constraint on $\nu = (\nu_\alpha)$ is $\nu_\alpha > 0$ and $\sum_{\alpha \in \{0, 1\}^K} \nu_\alpha = 1$. Therefore, we obtain the following probability mass function of the response vector $y$ under the commonly adopted local independence assumption (i.e., observed variables are conditionally independent given the latent),

$$\mathbb{P}(y = c \mid G, \theta, \nu) = \sum_{\alpha \in \{0, 1\}^K} \nu_\alpha \prod_{j=1}^{p} \prod_{k=1}^{K} \left[ \left( \theta_{c_j|1}^{(j)} \right)^{\alpha_k} \cdot \left( \theta_{c_j|0}^{(j)} \right)^{1-\alpha_k} \right]^{g_{j,k}},$$

where $c = (c_1, \ldots, c_p)^T \in \times_{j=1}^{p} [d]$ is an arbitrary response pattern. The name Binary Latent cliquE Star foreSt (BLESS) model is suggested as Equation (2) does not assume any conditional or marginal independence relations among latent variables a priori, and hence the graph among the latent can be viewed as a “clique” a priori in the graph terminology.

In real-world applications, the BLESS model can be useful in educational assessments, epidemiological diagnostic tests, and social science surveys, where the presence/absence of multiple latent characteristics are of interest and there are several observed proxies measuring each of them. For instance, in disease etiology in epidemiology (Wu et al., 2017), we can use each $\alpha_k$ to denote the presence/absence of a pathogen, and for each pathogen a few noisy diagnostic measures $y_j$’s are observed as the children variables of $\alpha_k$. See Section 5 for two real-world examples. In addition, our BLESS model is interestingly connected to a family of models used in causal discovery and machine learning, the pure-measurement models in Silva
Those are linear models of continuous variables, where the latent variables are connected in an acyclic causal graph; the commonality with the BLESS model is that each observed variable has at most one latent parent. The BLESS model can be thought of as a discrete analogue of such a pure-measurement model in Silva et al. (2006), and indeed more general in terms of the latent dependence structure. This is because we do not constrain the $\alpha_k$’s to follow a acyclic graph distribution but rather allow them to be arbitrarily dependent. Our identifiability conclusions always hold under this general setup.

### 2.2 Strict, Generic, and Local Identifiability

We first define strict identifiability in the context of the BLESS model. All the model parameters are included in the identifiability consideration, including the continuous parameters: the conditional probabilities $\theta = \{\theta_{c_i|1, c_j|1}^{(j)}\}$ and the proportions $\nu$; and the discrete measurement graph structure $G$.

**Definition 1 (Strict Identifiability).** The BLESS model is said to be strictly identifiable under certain conditions, if for valid parameters $(G, \theta, \nu)$ satisfying these conditions, the following equality holds if and only if $(G, \theta, \nu)$ and $(\overline{G}, \overline{\theta}, \overline{\nu})$ are identical up to a common permutation of $K$ latent variables:

$$P(y = c \mid G, \theta, \nu) = P(y = c \mid \overline{G}, \overline{\theta}, \overline{\nu}), \quad \forall c \in \times_{j=1}^{p}[d].$$

The statement of “identifiable up to latent variable permutation” in Definition 1 is an inevitable property of any latent variable model. We next define generic identifiability in the context of the BLESS model. Generic identifiability is a concept proposed and popularized by Allman et al. (2009). Given a graphical matrix $G$ and some valid continuous parameters $(\theta, \nu)$ under the BLESS model, define the following subset of the parameter space as

$$\mathcal{N}^G = \{(\theta, \nu) \text{ are associated with some graphical matrix } G : \exists (\overline{\theta}, \overline{\nu}) \text{ associated with some graphical matrix } \overline{G} \text{ such that } P(y \mid G, \theta, \nu) = P(y \mid \overline{G}, \overline{\theta}, \overline{\nu})\}.$$

**Definition 2 (Generic Identifiability).** Under a BLESS model, parameters $(G, \theta, \nu)$ are said to be generically identifiable under certain conditions, if for valid parameters $(G, \theta, \nu)$
satisfying these conditions, the set $N^G$ defined in (4) has measure zero with respect to the Lebesgue measure on the parameter space of $(\theta, \nu)$ under the identifiability conditions.

It is believed that generic identifiability often suffices for data analyses purposes (Allman et al., 2009). Finally, we define local identifiability of continuous parameters in the model.

**Definition 3** (Local Identifiability). Under a BLESS model, a continuous parameter $\mu$ (e.g., some entry of $\theta$ or $\nu$) is said to be locally identifiable, if there exists an open neighborhood $S$ of $\mu$ in the parameter space such that there does not exist any alternative parameter $\bar{\mu} \in S \setminus \{\mu\}$ leading to the same distribution of the response vector $y$.

The lack of local identifiability usually has severe consequences in practice, because in an arbitrarily small neighborhood of the true parameter, there exist infinitely many alternative parameters that give rise to the same observed distributions. This would render any estimation and inference conclusions invalid.

## 3 Main Theoretical Results

### 3.1 Theoretical Results of Generic Identifiability and Their Illustrations

In this subsection we will present sharp identifiability conditions and the blessing-of-dependence geometry for the BLESS model. The later Section 3.2 will provide an overview of the general algebraic proof technique used to derive the identifiability results. Throughout this work, assume $\nu_{\alpha} > 0$ for any latent pattern $\alpha \in \{0, 1\}^K$; i.e., all the possible binary latent patterns exist in the population with nonzero proportions. This is the only assumption imposed on the distribution of the latent variables, simply requiring the proportion parameters not to be on the boundary of the probability simplex. There is no assumption on whether or how the latent variables should depend on each other in the BLESS model.

It may be expected that each latent variable needs to have at least one observed child (i.e., $\sum_{j=1}^P g_{j,k} \geq 1$) to ensure identifiability of the BLESS model. What may not be expected at first is that such a condition is insufficient even for generic identifiability or local identifiability to hold. Our first conclusion below shows the condition that each latent variable has at least two observed children is necessary for generic identifiability or local identifiability.
**Proposition 1** (Necessary Condition for Generic Identifiability: \( \geq 2 \) children). The following two conclusions hold.

(a) If some binary latent variable has only one observed variable as child, then the model parameters are **not** generically identifiable and **not** locally identifiable.

(b) Specifically, suppose \( \alpha_k \) has only one observed \( y_j \) as child, then any of the \( \theta_{c|0}^{(j)} \) and \( \theta_{c|1}^{(j)} \) for \( c \in [d] \), and \( \nu_\alpha \) for \( \alpha \in \{0,1\}^K \) can not be generically or locally identifiable. In an arbitrarily small neighborhood of any of these parameters, there exist alternative parameters that lead to the same distribution of the observables as those given by the truth.

**Remark 1.** Proposition 1 also has an interesting implication on a seemingly unrelated problem: learning tree models from noisy data. Nikolakakis et al. (2021) considered learning hidden tree-structured Ising models, which essentially can be reformulated as the BLESS model where each node in the latent tree has exactly one child (its noisy observed proxy) and the responses are binary, i.e., \( G = I_K \) and \( d = 2 \). Nikolakakis et al. (2021) derived the sample complexity under the assumption that the noise level at each node is homogeneous and known. Proposition 1 implies that when the noise level is unknown and potentially heterogeneous, then these node-wise noise parameters (analogous to our \( \theta \)) are not even generically or locally identifiable, no matter what structure the tree graph among the latent nodes is.

The conclusion of “not even generically identifiable or locally identifiable” in Proposition 1 has quite severe consequences in parameter interpretation or estimation. There will be one-dimensional continuum of each of \( \theta_{c|0}^{(j)} \) and \( \theta_{c|1}^{(j)} \) for \( c \in [d] \), and \( \nu_\alpha \) for \( \alpha \in \{0,1\}^K \), that lead to the same probability mass function of the response vector \( y \). As revealed in part (b) of Proposition 1, the parameter space will have “flat regions” where identifiability is no hope, hence any statistical analysis in this scenario will be meaningless.

In Figure 2, we provide a numerical example to illustrate and corroborate Proposition 1. Consider \( p = 5 \) binary response variables, \( K = 3 \) binary latent variables, and a \( 5 \times 3 \) graphical matrix \( G = (100; 010; 001; 010; 001) \). This \( G \) indicates that the first latent variable \( \alpha_1 \) only has one observed child \( y_1 \), violating the necessary condition for generic identifiability in Proposition 1. In the left panel of Figure 2, the horizontal axis records nine continuous
parameters in the model, including one conditional probability $\theta_{1|1}^{(1)}$ and $2^K = 8$ proportion parameters for the binary latent pattern; the black solid line represents one set of true parameters, while the 150 colored lines represent those alternative parameters in a neighborhood of the truth constructed based on the proof of Proposition 1. To see the non-identifiablility, we calculate the probability mass function of the 5-dimensional binary response vector $y$, which has $2^p = 32$ entries, and plot it under the true and alternative parameters in the right panel of Figure 2. In particular, the horizontal axis in the plot presents the indices of the response patterns $c \in \{0, 1\}^5$, and the vertical axis presents the values of $\mathbb{P}(y = c | G, \theta, \nu)$, where the “+” symbols correspond to response probabilities given by the true parameters and the “○” represents those given by the 150 sets of alternative parameters. The marginal response probabilities of the observables given by all the alternative parameters perfectly equal those under the truth. This illustrates the severe consequence of lack of local identifiability.

![Figure 2: Illustrating Proposition 1, severe consequence of the lack of local identifiability.](image)

Figure 2: Illustrating Proposition 1, severe consequence of the lack of local identifiability. The $G_{5\times3} = (100; 010; 001; 010; 001)$. On the left panel, the black line represents the true set of parameters and each colored line corresponds to an alternative set of parameters. On the right panel, the marginal probability mass functions of the observed $y \in \{0, 1\}^5$ are plotted for all the parameter sets, “+” for the true set and circles “○” for alternative sets.

Now that each latent variable has to have $\geq 2$ observed children for generic identifiability to possibly hold, next we focus on this scenario. In fact, our next result shows that such a condition is sufficient for identifying discrete structure $G$ of the BLESS model. Such a result
is technically quite nontrivial, and in fact can not be derived using existing techniques such as Kruskal’s Theorem.

**Theorem 1** (Identifiability of the Latent-to-observed Star Forest $G$). In the BLESS model, if each latent variable has at least two observed variables as children (i.e., $\sum_{j=1}^{p} g_{j,k} \geq 2$), then the latent-to-observed star forest structure $G$ is identifiable.

We have the following main theorem on the identifiability of all the parameters in the BLESS model, which reveals the “blessing of dependence” phenomenon.

**Theorem 2** (Blessing of Latent Dependence and Generic Identifiability). In the BLESS model, suppose each latent variable has exactly two observed variables as children. The following conclusions hold.

(a) The model with parameters $(G, \theta, \nu)$ is generically identifiable.

(b) Specifically, any valid set of model parameters $(\theta, \nu)$ are identifiable if and only if the $K$ latent variables are not independent according to $\nu$.

We provide a numerical example to illustrate the blessing-of-dependence phenomenon and corroborate Theorem 2. Consider the BLESS model with each observed variable having $d = 3$ categories and $G = (I_2; I_2)^\top$. We first randomly generate $M = 100$ sets of true parameters of the BLESS model, from which we further generate the observed datasets. Given a fixed sample size $N = 10^4$, for each of the $M = 100$ parameter sets we further generate $L = 200$ independent datasets each with $N$ data points. We further use an EM algorithm (presented as Algorithm 1 in the Supplementary Material) to compute the maximum likelihood estimators (MLE) of the model parameters for each dataset; here we focus on estimating continuous parameters $(\theta, p)$ with $G$ fixed, because $G$ is guaranteed to be identifiable by Theorem 1. Ten random initializations are taken for the EM algorithm and we keep the one with the largest log likelihood value as the MLE. After collecting the MLEs, we calculate the Mean Squares Errors (MSEs) of continuous parameters calculated based on the 200 datasets for each of the 100 true parameter sets.

Figure 3 plots the 100 sets of values of the proportion parameters $\nu = (\nu_\alpha, \alpha \in \{0, 1\}^2)$ inside the latent probability simplex $\Delta^{2K-1} = \Delta^3$; such a simplex takes the shape of a
Figure 3: Corroborating Theorem 2. Latent probability simplex $\Delta^{2^2-1}$ for the proportion parameters of $\alpha = (\alpha_1, \alpha_2) \in \{0, 1\}^2$, where the saddle surface corresponds to the independence model of two latent variables $\alpha_1 \perp \perp \alpha_2$. Black balls correspond to those parameter sets which have the largest 20% MSEs across the 100 sets, while blue balls correspond to the remaining 80% parameter sets. MSEs are calculated based on sample size $N = 10^4$.

Figure 4: Illustrating Theorem 2. The $G_{6 \times 3} = (I_3; I_3)$. The true parameters $\nu_{\text{true}}$ falls on the independence surface of $\alpha_1 \perp \perp \alpha_{2:3}$. On the right panel, the marginal probability mass functions of the observed $y \in \{0, 1\}^5$ are plotted for all the parameter sets, “+” for the true set and circles “○” for 150 alternative sets.
polyhedron in three dimensions where $x$, $y$, $z$-axes correspond to $\nu_{00}, \nu_{01}, \nu_{11}$, respectively. For reference, we also plot the subset of the $\Delta^3$ that corresponds to the case of independent $\alpha_1$ and $\alpha_2$ inside the simplex. Such a subset is a smooth surface determined by $\nu_{00}\nu_{11} - \nu_{01}\nu_{10} = 0$, which can be equivalently written as $\nu_{00}\nu_{11} - \nu_{01}(1 - \nu_{00} - \nu_{11} - \nu_{01}) = 0$ in terms of the three coordinates $(\nu_{00}, \nu_{01}, \nu_{11})$. This subset takes the shape of a saddle surface embedded in the interior of the polyhedron, as shown in Figure 3. Each dataset is plotted as a solid ball inside the latent simplex. In particular, we plot those parameter sets with the largest 20% MSEs as black balls and plot the remaining parameters as blue balls. The two views shown in Figure 3 clearly show that the black balls are closer to the saddle surface of $\alpha_1 \perp \perp \alpha_2$. The simulation results demonstrate that the independence submodel of the latent variables defines a singular subset well within the interior of the parameter space (rather than on the boundary of it). Parameter estimation under the model becomes harder when true parameters are closer to this singular subset of $\alpha_1 \perp \perp \alpha_2$.

We next provide another example to illustrate the other part of Theorem 2 – that when the latent variables are indeed independent, each with two observed children, then the BLESS model parameters are unidentifiable. In Figure 4, we further use the proof of Theorem 2 to construct such indistinguishable sets of parameters. Specifically, the true proportion parameters $\nu^{\text{true}}$ are constructed such that it implies $\alpha_1 \perp \perp (\alpha_2, \alpha_3)$; in particular, this means the two subvectors $(\nu_{000}^{\text{true}}, \nu_{001}^{\text{true}}, \nu_{010}^{\text{true}}, \nu_{011}^{\text{true}})$ and $(\nu_{100}^{\text{true}}, \nu_{101}^{\text{true}}, \nu_{110}^{\text{true}}, \nu_{111}^{\text{true}})$ are linearly dependent; the true parameters are plotted with black “+”s on the left panel of Figure 4. We then follow the proof of Theorem 2 to construct 150 alternative sets of parameters and plot each set as a colored line with circles. Then similarly to that in Figure 2, we calculate the marginal response probabilities of the observed vector $y$ under the true and all the alternative parameter sets, respectively, and plot them in the right panel of Figure 4. These $2^L$-dimensional marginal probability vectors are exactly equal under all the parameter sets, confirming the nonidentifiability. Combining Figure 3 and Figure 4, we have demonstrated that in the boundary case where each latent variable has two observed children, the parameters are generically identifiable and will become nonidentifiable when the latent variables are independent. These observations corroborate Theorem 2.

We next present a more nuanced statement about identifiability implied by the proof of the main result Theorem 2. In the BLESS model, denote by $\text{Child}(\alpha_k) \mid \alpha_k$ the conditional
distribution of all the child variables of $\alpha_k$ given $\alpha_k$; hence $\text{Child}(\alpha_k) = \{y_j : g_{j,k} = 1\}$. Specifically, the parameters associated with $\text{Child}(\alpha_k) \mid \alpha_k$ are the following conditional probabilities:

$$\left\{\theta^{(j)} : y_j \in \text{Child}(\alpha_k)\right\} = \left\{\theta^{(j)}_{1:d|0}, \theta^{(j)}_{1:d|1} : g_{j,k} = 1\right\}. \quad (5)$$

Our proof of Theorem 2 gives the following fine-grained identifiability arguments regarding each latent variable.

**Corollary 1** (Blessing of Latent Dependence for Each Latent Variable). For each latent variable $\alpha_K$ that has exactly two observed variables as children, the following two statements are equivalent, in that (S1) is true if and only if (S2) is true.

(S1) $\alpha_k \not\perp \perp (\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_K)$ holds;

(S2) the parameters associated with the conditional distributions $\text{Child}(\alpha_k) \mid \alpha_k$ defined in (5) are identifiable.

**Remark 2.** We would like to emphasize that it is impossible to obtain the conclusions of Theorem 1, Theorem 2, and Corollary 1 by applying Kruskal’s theorem. In fact, the observed probability tensor $\Pi = (\pi_{c_1,\ldots,c_p})$ in the minimal generically identifiable case cannot be concatenated in any way as in Allman et al. (2009) to satisfy Kruskal’s rank conditions for unique three-way tensor decompositions. The proofs of Theorem 1 and Theorem 2 are in fact quite nontrivial. In Section 3.2 we will give an overview of the general algebraic technique used to prove these results.

It is useful to adopt a graphical perspective to our identifiability results of the BLESS model. Figure 5(a)-(b) provide graphical illustrations of generic identifiability conclusions and the blessing of dependence phenomenon. With $K = 5$ latent variables each having two observed variables as children (i.e., $G = (I_K; I_K)^\top$), the parameters corresponding to Figure 5(a) are identifiable due to the dependence indicated by the dotted edges between $\alpha_1, \ldots, \alpha_5$; while the parameters corresponding to Figure 5(b) are not identifiable due to the lack of dependence between $\alpha_1$ and $\alpha_{-1} := (\alpha_2, \ldots, \alpha_5)$. Such identifiability arguments guaranteed by Corollary 1 are of a very fine-grained nature, revealing that the dependence between a
Figure 5: CPTs refer to Conditional Probability Tables. All nodes are discrete random variables, with $\alpha_k \in \{0, 1\}$ latent and $y_j \in \{1, \ldots, d\}$ observed. The parameters corresponding to the dashed directed edges in (b) are unidentifiable, because $\alpha_1$ is independent of $\alpha_{2:5} = (\alpha_2, \alpha_3, \alpha_4, \alpha_5)$.

specific latent variable and the remaining ones exactly determines the identifiability of the conditional probability tables given this particular latent variable.

The identifiability results in Theorem 1 and Theorem 2 yield the following observations. First, a notable fact is that our proof reveals that the blessing of latent dependence always holds regardless of the sign of the dependence. Either positive dependence or negative dependence helps deliver model identifiability. Second, the easiest scenario for the star forest structure $G$ to be identifiable seems to be the hardest one for the continuous parameters to
be identifiable. To see this, consider the extreme case where all the $K$ latent variables are perfectly dependent\(^1\), then the star forest structure cannot be recovered, because it is impossible to tell apart which observed variables are children of which latent ones. Generally, the more independent the latent variables are, the easier it should be to identify the measurement graph. On the other hand, however, according to our conclusions in Theorem 2 and Corollary 1, having the latent variables independent is the hardest, and in fact impossible, scenario for the continuous parameters $\theta$ and $\nu$ to be identifiable when each $\alpha_k$ has two children. This perhaps counterintuitive phenomenon shows the complexity and surprising geometry of discrete graphical models with latent variables.

Interestingly, the case of each latent variable having two children forms the exact boundary for the blessing of dependence to play a role. In fact, as long as each latent variable has at least three observed variables as children, the Kruskal’s theorem (Kruskal, 1977) on the uniqueness of three-way tensor decompositions can “kick in” to guarantee identifiability. In particular, we can use an argument similar to that in Allman et al. (2009) to establish this conclusion, by concatenating certain observed variables into groups and transforming the underlying $p$-way probability tensor into a three-way tensor. The following proposition formalizes this statement.

**Proposition 2** (Kruskal’s Theorem Kicks in for the $\geq 3$ Children Case). Consider model (2) with (1) satisfied. If each latent variable has three or more observed variables as children (i.e., $\sum_{j=1}^{p} g_{j,k} \geq 3$), then the model is always strictly identifiable, regardless of the dependence between latent variables.

**Example 1.** Consider the BLESS model with $K = 3$, $p = 7$ and the following $7 \times 3$ graphical matrix

$$G = \begin{pmatrix} I_3 \\ I_3 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then by Theorem 1, the $G$ matrix itself is identifiable from the joint distribution of the ob-

\(^1\)Note that in this extreme case, many latent patterns $\alpha$’s will have population proportions zero and hence does not satisfy our only assumption on $\nu$. Therefore the fact that $G$ is unidentifiable in this extreme case does not contradict our result on the identifiability of $G$ in Theorem 1.
served variables. And by Theorem 2, the continuous parameters are generically identifiable. Further, since $\alpha_1$ has three children with $\text{Child}(\alpha_1) = \{y_1, y_4, y_7\}$, by Proposition 2, the conditional probability tables $\theta^{(1)}, \theta^{(4)}, \theta^{(7)}$ are strictly identifiable regardless of the dependence between the variable $\alpha_1$ and the other latent variables $(\alpha_1, \alpha_2)$. By Corollary 1, since $\alpha_2$ has two children $\text{Child}(\alpha_2) = \{y_2, y_5\}$, the $\theta^{(2)}, \theta^{(5)}$ are identifiable if and only if $\alpha_2 \not\perp \perp (\alpha_1, \alpha_3)$. Similarly, parameters $\theta^{(3)}, \theta^{(6)}$ are identifiable if and only if $\alpha_3 \not\perp \perp (\alpha_1, \alpha_2)$.

Summarizing all the above conclusions in this section, we have the following conclusions.

**Corollary 2.** Consider the BLESS model. The following statements hold.

(a) The condition that each binary latent variable has $\geq 2$ observed variables as children is **necessary and sufficient** for the generic identifiability of the model parameters.

(b) The condition that each binary latent variable has $\geq 3$ observed variables as children is **necessary and sufficient** for the strict identifiability of the model parameters.

Corollary 2 describes the minimal conditions for strict identifiability and those for generic identifiability of the BLESS model, respectively. The conclusions in Corollary 2 are immediate consequences of Theorem 2 and Proposition 2. It is worth noting that both the minimal conditions for strict identifiability and those for generic identifiability only concern the discrete structure in the model – the measurement graph $G$, but not on the specific values of the continuous parameters $\theta$ or $\nu$. Therefore, these identifiability conditions as graphical criteria are easily checkable in practice.

The blessing of dependence phenomenon when each latent variable has two children has nontrivial connections to and implications on the uniqueness of matrix and tensor decompositions. *On one hand*, if each latent variable has two children with $p = 2K$ and $y_k, y_{K+k}$ are children of $\alpha_k$ for each $k \in [K]$, then we can group the first $y_1, \ldots, y_K$ and define a surrogate categorical variable $Z_1 = (y_1, \ldots, y_K) \in [d]^K$ with $d^K$ latent states, and similarly group the $y_{K+1}, \ldots, y_{2K}$ to define $Z_2 \in [d]^K$. The joint contingency table of $(Z_1, Z_2)$ can then be expressed as a two-way table of size $d^K \times d^K$, where each entry in the table corresponds to the probability of a response pattern of the original vector $y$ and all these probabilities sum up to one. This setting can be considered as a reduced rank model for two-way contingency table (matrix) (De Leeuw and Van der Heijden, 1991), where the rank of the matrix is
$|\{0,1\}^K| = 2^K$, equal to the number of states the latent vector $\alpha$ can take. It is well-known that such a matrix factorization generally can not be unique. *On the other hand*, if each latent variable has three children with $p = 3K$ and $y_{k}, y_{K+k}, y_{2K+k}$ being children of $\alpha_k$, then we can define $Z_1 = (y_1,\ldots,y_K)$, $Z_2 = (y_{K+1},\ldots,y_{2K})$, and $Z_3 = (y_{2K+1},\ldots,y_{3K})$. The joint contingency table of $Z_1, Z_2, Z_3$ is a three-way tensor. Due to the conditional independence of $Z_1, Z_2, Z_3$ given the latent $\alpha$, such a tensor has a CP decomposition (Kolda and Bader, 2009) of rank $2^K$. By Kruskal’s Theorem, this three-way decomposition is identifiable under mild conditions. Our results reveal that between the well-known unidentifiable matrix factorization and well-known identifiable tensor decomposition, there is a special middle ground where the dependence between multiple binary latent variables helps restore identifiability.

### 3.2 Overview of the Proof Techniques and Its Usefulness

In this subsection we provide an overview of the general proof techniques used to derive the identifiability results. For the ease of understanding, we next describe the technique in the context of multidimensional binary latent variables; we will later explain that these techniques are generally applicable to discrete models with latent and graphical components.

With $K$ binary latent variables, define the binary vector representations of integers $1,\ldots,2^K$ by $\alpha_1, \alpha_2, \ldots, \alpha_{2^K}$; that is, for a $K$-dimensional vector $v = (2^K - 1, 2^K - 2, \ldots, 2^0)^\top$ there is

$$\alpha_\ell^\top v = \ell + 1, \quad \ell = 1, 2, \ldots, 2^K.$$ 

Each $\alpha_\ell$ represents a binary latent pattern describing the presence or absence of the $K$ latent variables and $\{\alpha_1, \ldots, \alpha_{2^K}\} = \{0,1\}^K$. With $p$ discrete observed variables $y_1, \ldots, y_p$, generally denote the conditional distribution of each $y_j$ given latent pattern $\alpha_\ell$ by

$$\theta^{(j)}_{c|a_{\ell}} = \mathbb{P}(y_j = c \mid a = \alpha_\ell), \quad j \in [p], \; c \in [d], \; \ell \in [2^K].$$

Note that under the BLESS model, the $\theta^{(j)}_{c|a_{\ell}}$ is a reparametrization of the probabilities $\theta^{(j)}_{c|1}$ and $\theta^{(j)}_{c|0}$. According to the star-forest measurement graph structure, whether $\theta^{(j)}_{c|a_{\ell}}$ equals $\theta^{(j)}_{c|1}$ or $\theta^{(j)}_{c|0}$ depends only on whether or not the pattern $\alpha_\ell$ possesses the latent parent of $y_j$. Mathematically, since vector $g_j$ summarizes the parent variable information of $y_j$, we have
that

\[
\theta_{c_l|\alpha_t}^{(j)} = \begin{cases} 
\theta_{c_1|\alpha}^{(j)}, & \text{if } \alpha_{t,k} = 1 \text{ for the } k \text{ where } g_{j,k} = 1; \\
\theta_{c_0|\alpha}^{(j)}, & \text{if } \alpha_{t,k} = 0 \text{ for the } k \text{ where } g_{j,k} = 1.
\end{cases}
\]

(6)

In the above expression, the \(\alpha_{t,k}\) denotes the \(k\)th entry of the binary pattern \(\alpha_t\). For each observed variable index \(j \in [p]\), define a \(d \times 2^K\) matrix \(\Phi^{(j)}\) as

\[
\Phi^{(j)} = \begin{pmatrix}
P(y_j = 1 | a = \alpha_1) & \cdots & P(y_j = 1 | a = \alpha_{2^K}) \\
\vdots & \ddots & \vdots \\
P(y_j = d | a = \alpha_1) & \cdots & P(y_j = d | a = \alpha_{2^K})
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\theta_{c_1|\alpha_1}^{(j)} & \cdots & \theta_{c_1|\alpha_{2^K}}^{(j)} \\
\vdots & \ddots & \vdots \\
\theta_{c_d|\alpha_1}^{(j)} & \cdots & \theta_{c_d|\alpha_{2^K}}^{(j)}
\end{pmatrix},
\]

then \(\Phi^{(j)}\) is the conditional probability table of variable \(y_j\) given \(2^K\) latent patterns. Each column of \(\Phi^{(j)}\) is indexed by a pattern \(\alpha_t\) and gives the conditional distribution of variable \(y_j\) given the latent pattern \(\alpha_t\). Note that many entries in \(\Phi^{(j)}\) are equal due to (6); we deliberately choose this overparameterized matrix notation to facilitate further tensor algebra. The equality of the many parameters in each \(\Phi^{(j)}\) will later be carefully exploited when examining identifiability conditions.

Denote by \(\otimes\) the Kronecker product of matrices and denote by \(\odot\) the Khatri-Rao product (Kolda and Bader, 2009). The Khatri-Rao product is a column-wise Kronecker product of matrices, and for two matrices with the same number of columns \(A = (a_{i,j}) = (a_{i,1} | \cdots | a_{i,k}) \in \mathbb{R}^{n \times k}\), \(B = (b_{i,j}) = (b_{i,1} | \cdots | b_{i,k}) \in \mathbb{R}^{\ell \times k}\), their Khatri-Rao product \(A \odot B \in \mathbb{R}^{n\ell \times k}\) still has the same number of columns and can be written as

\[
A \odot B = \begin{pmatrix} a_{i,1} \otimes b_{i,1} & \cdots & a_{i,k} \otimes b_{i,k} \end{pmatrix}.
\]
Under the considered model, all the $d^p$ marginal response probabilities form a $p$-way tensor

$$\Pi = (\pi_{c_1, \ldots, c_p}), \quad c_j \in [d],$$

where each entry $\pi_{c_1, \ldots, c_p} = \mathbb{P}(y_1 = c_1, \ldots, y_p = c_p \mid \text{star-forest structure and parameters})$ denotes the marginal probability of observing the response pattern $y = c$ under the latent variable model. With the above notation, the probability mass function (PMF) of vector $y$ under the BLESS model in (2) can be equivalently written as

$$\text{vec}(\Pi) = \left( \bigotimes_{j=1}^{p} \Phi^{(j)} \right) \cdot \nu,$$  \quad (7)

where $\text{vec}(\Pi)$ denotes the vectorization of the tensor $\Pi$ into a vector of length $d^p$. The Khatri-Rao product of $\Phi^{(j)}$ in the above display results from the basic local independence assumption in (2). We next state a useful technical lemma. The following lemma characterizes a fundamental property of the transformations of Khatri-Rao product of matrices.

**Lemma 1.** Consider an arbitrary set of conditional probability tables $\{\Phi^{(j)} : j \in [p]\}$, where $\Phi^{(j)}$ has size $d_j \times 2^K$ with each column summing to one. Given any set of vectors $\{\Delta_j : j \in [p]\}$ with $\Delta_j = (\Delta_{j,1}, \ldots, \Delta_{j,d_j-1}, 0)^\top \in \mathbb{R}^{d_j \times 1}$, there exists a $\prod_{j=1}^{p} d_j \times \prod_{j=1}^{p} d_j$ invertible matrix $B := B(\{\Delta_j : j \in [p]\})$ determined entirely by $\{\Delta_j : j \in [p]\}$ such that

$$\bigotimes_{j\in[p]} \left( \Phi^{(j)} - \Delta_j \cdot 1_{2^K}^\top \right) = B(\{\Delta_j : j \in [p]\}) \cdot \left( \bigotimes_{j\in[p]} \Phi^{(j)} \right),$$ \quad (8)

where $\Delta_j \cdot 1_{2^K}^\top$ is a $d_j \times 2^K$ matrix, of the same dimension as $\Phi^{(j)}$.

In addition, replacing the index $j \in [p]$ in (8) by $j \in S$ where $S$ is an arbitrary subset of $[p]$ on both hand sides still makes the equality holds.

Lemma 1 covers as special case a result in Xu (2017) for restricted latent class models with binary responses. Instead of exclusively considering moments of binary responses as Xu (2017), our Lemma 1 here characterizes a general algebraic property of Khatri-Rao products of conditional probability tables of multivariate categorical data. This property together with the model formulation in (7) will enable us to exert various transformations on the
model parameters to investigate their identifiability. We provide a proof of Lemma 1 below, because it is concise and delivers an insight into our technique’s usefulness.

Proof of Lemma 1. Consider an arbitrary subset $S \in [p]$. First note that the sum of all the entries in each column of $\Phi^{(j)}$ is one because each column vector is a conditional probability distribution of $y_j$ given a particular latent pattern. Therefore with $\Delta_j = (\Delta_{j,1}, \ldots, \Delta_{j,d-1}, 0)^T$, we have

$$\Phi^{(j)} - \Delta_j \cdot \mathbf{1}_{2K}^T = \begin{pmatrix} 
\theta_{1|\alpha_1}^{(j)} - \Delta_{j,1} & \cdots & \theta_{1|\alpha_{2K}}^{(j)} - \Delta_{j,1} \\
\vdots & \ddots & \vdots \\
\theta_{d-1|\alpha_1}^{(j)} - \Delta_{j,d-1} & \cdots & \theta_{d-1|\alpha_{2K}}^{(j)} - \Delta_{j,d-1} \\
\theta_{d|\alpha_1}^{(j)} & \cdots & \theta_{d|\alpha_{2K}}^{(j)} 
\end{pmatrix} - \begin{pmatrix} 
\begin{pmatrix} 
\theta_{1|\alpha_1}^{(j)} \\
\vdots \\
\theta_{d-1|\alpha_1}^{(j)} \\
\theta_{d|\alpha_1}^{(j)} 
\end{pmatrix} \\
\begin{pmatrix} 
\theta_{1|\alpha_{2K}}^{(j)} \\
\vdots \\
\theta_{d-1|\alpha_{2K}}^{(j)} \\
\theta_{d|\alpha_{2K}}^{(j)} 
\end{pmatrix} 
\end{pmatrix} = \begin{pmatrix} 
1 & 0 & \cdots & 0 & -\Delta_{j,1} \\
0 & 1 & \cdots & 0 & -\Delta_{j,2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -\Delta_{j,d-1} \\
-1 & -1 & \cdots & -1 & 1 
\end{pmatrix} \cdot \begin{pmatrix} 
\begin{pmatrix} 
\theta_{1|\alpha_1}^{(j)} \\
\vdots \\
\theta_{d-1|\alpha_1}^{(j)} \\
\theta_{d|\alpha_1}^{(j)} 
\end{pmatrix} \\
\begin{pmatrix} 
\theta_{1|\alpha_{2K}}^{(j)} \\
\vdots \\
\theta_{d-1|\alpha_{2K}}^{(j)} \\
\theta_{d|\alpha_{2K}}^{(j)} 
\end{pmatrix} 
\end{pmatrix} = \begin{pmatrix} 
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
-1 & -1 & \cdots & -1 & 1 
\end{pmatrix} \cdot \Phi^{(j)}$$

$$= \tilde{\Delta}_j C \Phi^{(j)}.$$
also has full rank $2^K$. Then
\[
\bigcirc_{j \in S} \begin{pmatrix} \Phi^{(j)} - \Delta_j \cdot 1_{2^K}^\top \end{pmatrix} = \bigcirc_{j \in S} \left( \tilde{\Delta}_j C \Phi^{(j)} \right) = \bigotimes_{j \in S} (\tilde{\Delta}_j C) \cdot \bigodot_{j \in S} \Phi^{(j)},
\]
where the last equality above follows from basic properties of the Kronecker and Khatri-Rao products and can be verified by checking corresponding entries in the products. Now define
\[
B (\{ \Delta_j : j \in S \}) := \bigotimes_{j \in S} (\tilde{\Delta}_j C),
\]
then $B (\{ \Delta_j : j \in S \})$ is a $d^{[S]} \times d^{[S]}$ matrix and it is invertible because it is the Kronecker product of $|S|$ invertible matrices $\tilde{\Delta}_j C$. This proves Lemma 1.

Recall that many entries in $\Phi^{(j)}$ are constrained equal under the graphical matrix $G$; in fact, the $\Phi^{(j)}$ is entirely determined by $G$ and $\theta$ and also the structure of $G$ and $\theta$ can be read off given the $\Phi^{(j)}$. Now suppose an alternative graphical matrix $\tilde{G} \in \{0,1\}^{p \times K}$ and some associated alternative parameters $(\tilde{\theta}, \tilde{\nu})$ lead to the same distribution as $(G, \theta, \nu)$. Then by (7), the following system of $d^{[S]}$ equations about the alternative parameters $\Phi$ and $\nu$ must hold for an arbitrary subset $S \subseteq [p]$,
\[
\left( \bigodot_{j \in S} \Phi^{(j)} \right) \cdot \nu = \left( \bigotimes_{j \in S} \Phi^{(j)} \right) \cdot \nu.
\]
Our goal is to study under what conditions on the true parameters, the alternative $(\tilde{G}, \tilde{\theta}, \tilde{\nu})$ must be identical to the true $(G, \theta, \nu)$. By Lemma 1, for arbitrary $\{ \Delta_j \}$, we have
\[
\left( \bigodot_{j \in S} \Phi^{(j)} - \Delta_j \cdot 1_{2^K}^\top \right) \cdot \nu = B (\{ \Delta_j : j \in S \}) \cdot \left( \bigotimes_{j \in S} \Phi^{(j)} \right) \cdot \nu
= B (\{ \Delta_j : j \in S \}) \cdot \left( \bigotimes_{j \in S} \Phi^{(j)} \right) \cdot \nu
= \left( \bigotimes_{j \in S} \Phi^{(j)} - \Delta_j \cdot 1_{2^K}^\top \right) \cdot \nu.
\]
(9)

We next give a high-level idea of our proof procedure. Eq. (9) will be frequently invoked for
various subsets $S \subseteq [p]$ when deriving the identifiability results. For example, suppose we want to investigate whether a specific parameter $\theta^{(j)}_{c|\alpha_\ell}$ is identifiable under certain conditions. Exploiting the fact that $G$ induces many equality constraints on entries of $\Phi^{(j)}$, we will construct a set of vectors $\{\Delta_j; j \in S\}$, which usually has the particular $\bar{\theta}^{(j)}_{c|\alpha_\ell}$ as an entry. These vectors $\{\Delta_j; j \in S\}$ are purposefully constructed such that the right hand side of Eq. (9) = 0 for some polynomial equation out of the $\prod_{j \in S} d_j$ ones. This implies a polynomial involving parameters $(G, \theta, \nu)$ and the constructed vectors $\{\Delta_j; j \in S\}$ is equal to zero. We will then carefully inspect under what conditions this equation gives $\theta^{(j)}_{c|\alpha_\ell}$’s identifiability; namely, inspect whether $\theta^{(j)}_{c|\alpha_\ell} = \bar{\theta}^{(j)}_{c|\alpha_\ell}$ must hold under the considered conditions. We emphasize here that our algebraic technique described above can be generally useful beyond the BLESS model. Essentially, our proof technique exploits the following two key model properties. First, observed variables are conditionally independent given the (potentially multiple) latent variables. This property makes it possible to write the joint distribution of the observed variables as the product of (a) Khatri-Rao product of individual conditional probability tables and (b) the vector of the joint probability mass function of latent variables. Second, there exist rich graphical structures involving the latent variables and observed variables. The graph will induce many equality constraints on the conditional probability table $\Phi^{(j)}$ of each observed variable given the configurations of the latent. The first property above about conditional independence is an extremely prevailing assumption adopted in many other latent variable models, and it is often called “local independence” in the literature. The second property above about graph-induced constraints is also frequently encountered across various directed and undirected graphical models (Lauritzen, 1996). Because of these two facts, we expect our techniques will be generally useful to find identifiability conditions for other complicated discrete models with multidimensional latent and graphical structures, e.g., discrete Bayesian networks with latent variables with application to causal inference (Allman et al., 2015; Mealli et al., 2016), mixed membership models (Erosheva et al., 2007), and overlapping community models for networks (Todeschini et al., 2020).
3.3 Discussing Connections to and Differences from Related Works

It is worth connecting the BLESS model to discrete Latent Tree Models (LTMs; Choi et al., 2011; Mourad et al., 2013), which are popular tools in machine learning and have applications in phylogenetics in evolutionary biology. Some deep results about the geometry and statistical properties of LTMs are uncovered in Zwiernik and Smith (2012), Zwiernik (2016), and Shiers et al. (2016). Conceptually, the BLESS model is more general than LTMs because in the former, the latent variables can have entirely flexible and arbitrarily complex dependence structure according to the definition in Eq. (2) (also implied by the word “clique” in the name of the BLESS model). Namely, the BLESS model only requires the latent-to-observed graph to be a tree and the latent graph can be a general clique; in contrast, LTMs require the entire graph among all the latent and observed variables to be a tree. As a result, the identifiability and geometry of the BLESS model are more complicated than those of the LTMs. Geometry and identifiability of Bayesian networks with hidden variables have also been investigated in Settimi and Smith (2000), Allman et al. (2015) and Anandkumar et al. (2013). But these works often either consider a small number of variables, or employ certain specific (rather than entirely flexible) assumptions on the dependence of latent variables.

Notably, in real-world applications in education and psychology, the aforementioned formulation of arbitrarily dependent latent variables has been widely employed in an emerging family of diagnostic models (e.g., Chen et al., 2015; Xu and Shang, 2018; Gu and Xu, 2021a; von Davier and Lee, 2019). This is because the binary latent variables in those applications have semantic meanings such as specific skills or mental disorders, and it is usually unsuitable to restrict the dependence graph between these latent constructs to be a tree. Rather, the latent variables may exhibit quite rich dependencies because of the complicated cognitive processes underlying learning or behaviors. Being able to derive sharp identifiability results without assuming any specific dependence structure among latent variables shows the power of the general algebraic technique we employ in this work.

A generic identifiability statement related to our work appeared in Gu and Xu (2021b) in the form of a small toy example for the aforementioned cognitive diagnostic models. More specifically, these are models where test items are designed to measure the presence/absence of multiple latent skills and binary item responses of correct/wrong answers are observed for
each subject. In the special case with two binary latent skills each measured by two binary observed variables, Gu and Xu (2021b) proved the parameters are identifiable if and only if the two latent variables are not independent. In this work, we investigate the fully general case of the BLESS model where there are (a) an arbitrary number of binary latent variables, (b) arbitrary dependence between these variables, and (c) the observed variables have an arbitrary number of categories. Under this general setup, we characterize a complete picture of the generic identifiability phenomenon with respect to the latent dependence in Section 3.1.

4 Statistical Hypothesis Test of Identifiability in the Boundary Case

Consider the minimal conditions for generic identifiability of the BLESS model, where certain (all or a subset of) latent variables have only two children. In this boundary scenario, a natural question of interest is whether one can decide whether the parameters are identifiable or not. To this end, it would be desirable to develop a formal statistical hypothesis test of identifiability. Our identifiability theory of blessing of dependence indeed provides a basis for such a simple testing approach. Under the star-forest measurement graph in the BLESS model, we have the following proposition.

Proposition 3. Under the BLESS model defined in (2), consider two different latent variables \( \alpha_{k_1} \) and \( \alpha_{k_2} \). The two groups of observed variables \( \{ y_j = c_j : g_{j,k_1} = 1 \} \) and \( \{ y_m = c_m : g_{m,k_2} = 1 \} \) are independent if and only if \( \alpha_{k_1} \) and \( \alpha_{k_2} \) are independent.

Proposition 3 states that under the BLESS model, the dependence/independence of latent variables is exactly reflected in the dependence/independence of their observed proxies (i.e., observed children variables). This fact is apparent from the graphical representation of the BLESS model in Figure 5; it can also be formally proved using the model definition in (2). A nice implication of Theorem 2 and Proposition 3 is that, we can test the marginal dependence between certain observed variables to determine model identifiability, before even trying to fit a potentially unidentifiable model to data.
Formally, in the boundary case (i.e., under minimal conditions for generic identifiability) where some latent variable $\alpha_k$ only has two observed children, if one wishes to test the following hypothesis

$$H_{0k} : \text{Parameters associated with } \text{Child}(\alpha_k) \mid \alpha_k \text{ are not identifiable},$$

then it is equivalent to testing the hypothesis $H'_{0k} : \alpha_k \perp \perp \alpha_{-k}$. Further, to test $H'_{0k}$ it suffices to test the marginal independence between the following observed variables,

$$H'_{0k} : \text{Child}(\alpha_k) \perp \perp \text{Child}(\alpha_{-k}).$$

Since Child$(\alpha_k)$ and Child$(\alpha_{-k})$ are fully observed given the measurement graph, the above hypothesis $H'_{0k}$ can be easily tested. Note that Child$(\alpha_k)$ can be regarded as a categorical variable with $d^{\text{Child}(\alpha_k)}$ categories and that Child$(\alpha_{-k})$ can be regarded as another categorical variable with $d^{\text{Child}(\alpha_{-k})}$ categories. So the simple $\chi^2$ test of independence between two categorical variables can be employed for testing $H'_{0k}$. If the null hypothesis of independence is not rejected, then caution is needed in applying the BLESS model because some parameters may not be identifiable. If, however, the hypothesis of independence is rejected, then this is statistical evidence supporting the identifiability of the BLESS model. In this case one can go on to fit the model to data, interpret the estimated parameters, and conduct further statistical analysis. In fact, if Child$(\alpha_{-k})$ consists of many observed variables, one can start with a small subset $S \subseteq \text{Child}(\alpha_{-k})$ and testing whether Child$(\alpha_k) \perp \perp S$; the rejection of this hypothesis would already provide evidence for identifiability of parameters (see Section 5.1 for such an example). We also point out that if multiple latent variables $\alpha_{k1}, \ldots, \alpha_{km}$ each has only two observed children, then one can test the $m$ hypotheses simultaneously

$$\{H'_{0k} : \text{Child}(\alpha_k) \perp \perp \text{Child}(\alpha_{-k}); \ k = 1, \ldots, m\},$$

and then use the Bonferroni correction to reach the final conclusion about the overall model identifiability.

**Example 2.** Continue to consider Example 1 where $G_{7 \times 3} = (I_3; \ I_3; \ 1 \ 0 \ 0)^\top$. Recall that $\{\theta^{(2)}, \theta^{(5)}\}$ are identifiable if and only if $\alpha_2 \not\perp \perp (\alpha_1, \alpha_3)$, and $\{\theta^{(3)}, \theta^{(6)}\}$ are identifiable if and
only if $\alpha_3 \not\perp (\alpha_1, \alpha_2)$. In order to test the hypothesis

$$H_{01} : \text{Parameters associated with } \text{Child}(\alpha_2) \mid \alpha_2 \ (\text{i.e. } \theta^{(2)}, \theta^{(5)}) \text{ are not identifiable,}$$

it suffices to test $H'_{01} : \alpha_2 \perp (\alpha_1, \alpha_3)$. Because of the form of the $G$ matrix, the test $H'_{01}$ of latent independence can be further reduced to the test of the following hypothesis of the observed variables,

$$H''_{01} : (y_2, y_5) \perp (y_1, y_3, y_4, y_6).$$

Similarly, in order to test

$$H_{02} : \text{Parameters associated with } \text{Child}(\alpha_3) \mid \alpha_3 \ (\text{i.e. } \theta^{(3)}, \theta^{(6)}) \text{ are not identifiable,}$$

it suffices to test the following hypothesis about the observed variables

$$H''_{02} : (y_3, y_6) \perp (y_1, y_2, y_4, y_5).$$

The tests of $H''_{01}$ and $H''_{02}$ can be carried out simply by testing the dependence between two concatenated categorical variables, one with $d^2$ categories and the other with $d^4$ categories.

Note that our hypothesis test of identifiability is performed without fitting the BLESS model to data, and can serve as a first-step sanity check in real data analysis. In a similar spirit but for a different purpose when studying the Gaussian Latent Tree Models, Shiers et al. (2016) proposed to test certain covariance structures of variables to determine the goodness of fit before fitting the model to data. To the author’s best knowledge, there has not been previous formal statistical approaches to directly testing the identifiability of multidimensional latent variable models. Our test of identifiability of the BLESS model is enabled by the discovery of the nontrivial blessing of dependence phenomenon and may inspire future relevant hypothesis testing approaches in other latent variable models.
5 Examples of Real-world Applications

This section presents two real-world examples where our new theory can bring new insights potentially – one in educational assessment and the other in social science surveys.

5.1 Educational assessment example

The Trends in International Mathematics and Science Study (TIMSS) is a series of international assessments of the mathematics and science knowledge of middle school students around the world. TIMSS assesses fourth and eighth grade students and it has been held every four years since 1995 in over 50 countries. The so-called cognitive diagnosis models have been used to analyze a subset of the Austrian TIMSS 2011 data in George and Robitzsch (2015); this dataset is available in the R package \texttt{CDM}. The dataset involves fourth grade students’ correct/wrong (binary) responses to a set of TIMSS questions in mathematics. According to psychometricians, these questions were designed to measure the presence/absence (binary) statuses of $K = 3$ content-based latent skills of students: $(\alpha_1)$ Data, $(\alpha_2)$ Geometry, and $(\alpha_3)$ Numbers. Each test question targets exactly one content-based skill, which means the latent-to-observed measurement graph satisfies the assumption of the BLESS model.

This original Austrian TIMSS dataset in the \texttt{CDM} package contains 1010 students’ responses to a total number of 47 questions but has many missing data, because the 47 items were divided up into three booklets and only two of the three booklets are presented to each student; such missingness is common to large-scale educational assessments (George and Robitzsch, 2015). To avoid dealing with the missing data issue in our example of identifiability considerations, here we focus on the first booklet containing the first $p = 21$ questions, and consider the $N = 341$ students who answered all these 21 questions. Table 1 summarizes the dependence of these 21 questions on the three underlying latent skills, i.e., the $G$ matrix structure in our notation, which is also provided in the R package \texttt{CDM}.

Table 1 shows that the first skill “Data” is measured by only two questions (questions 20 and 21), hence satisfying the minimal conditions for generic identifiability. So according to our new results, whether the model parameters are identifiable would depend on whether there exists underlying dependence between latent variables. We carry out a hypothesis test
Table 1: Educational assessment example of the Austrian TIMSS 2011 data. Latent-to-observed measurement graph structure between the first \( p = 21 \) questions and \( K = 3 \) content-based latent skills, constructed using the information available in the R package CDM.

<table>
<thead>
<tr>
<th>Content-based latent skill</th>
<th>Indices of questions that measure the skill</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 ) Data</td>
<td>20, 21</td>
</tr>
<tr>
<td>( \alpha_2 ) Geometry</td>
<td>7, 8, 16, 17, 18, 19</td>
</tr>
<tr>
<td>( \alpha_3 ) Numbers</td>
<td>1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15</td>
</tr>
</tbody>
</table>

of identifiability of the BLESS model. In particular, we want to test

\[ H_{0,\text{Data}} : \] The latent skill “Data” is independent of the two remaining skills “Geometry” and “Numbers”;

and based on the \( \mathbf{G} \) matrix structure in Table 1, we can test whether the questions targeting the “Data” skill are independent with those targeting other two skills. In particular, here we consider all the two-question-combinations consisting of one question targeting “Geometry” and one question targeting “Numbers”, and then test whether this combination of questions are independent of those two “Data” questions; namely, we test

\[ H_{0,\text{Data}}^{j_1,j_2} : (y_{20}, y_{21}) \text{ are independent of } (y_{j_1}, y_{j_2}), \quad j_1 \text{ targets Geometry, } j_2 \text{ targets Numbers.} \]

Using the standard \( \chi^2 \) test of independence between two categorical variables each of \( 2^2 = 4 \) categories, each test statistic under the null hypothesis \( H_{0,\text{Data}}^{j_1,j_2} \) asymptotically follows the \( \chi^2 \) distribution with \( df = (2^2-1)(2^2-1) = 9 \) degrees of freedom. Out of the \( 6 \times 13 = 78 \) such test statistics, we found 73 of them are greater than the 95\% quantile of the reference distribution \( \chi^2(df, 0.95) = 16.92 \), where we reject the null hypothesis of independence between \((y_{20}, y_{21})\) and \((y_{j_1}, y_{j_2})\). We point out that the rejection of any of these tests \( H_{0,\text{Data}}^{j_1,j_2} \) already indicates one should reject the original null \( H_{0,\text{Data}} \). Thanks to the blessing of dependence theory we have established, the test results provide statistical evidence to reject the original null hypothesis of non-identifiability, and hence support the identifiability of model parameters. This provides a statistical conclusion of identifiability for the first time in such applications in educational cognitive diagnosis modeling.
5.2 Prevention science survey example

An influential paper in prevention science Lanza and Rhoades (2013) used the latent class model (LCM; with a unidimensional latent variable) to analyse the treatment effects on different latent subgroups, and illustrated the method using a dataset extracted from the National Longitudinal Survey of Adolescent Health (NLSAH). Observed data for each subject are $p = 6$ dichotomized characteristics: household poverty; single-parent status; peer cigarette use; peer alcohol use; neighborhood unemployment; and neighborhood poverty. These observables actually measure three risks, with the first two measuring ($\alpha_1$) *household risk*, the middle two measuring ($\alpha_2$) *peer risk*, and the last two measuring ($\alpha_3$) *neighborhood risk*. According to the estimated conditional probability tables of the observed variables given the five latent classes, Lanza and Rhoades (2013) interpreted the latent classes as (a) Overall low risk, (b) Peer risk, (c) Household & neighborhood (economic) risk, (d) Household & peer risk, and (e) Overall high (multicontext) risk. Interestingly, we note that the analysis in Lanza and Rhoades (2013) lends itself to a reformulation using the BLESS model, and we argue that such a reformulation provides an interpretable graphical modeling alternative to plain latent class analysis. Specifically, if viewing the three underlying risks as three latent variables, then the latent-to-observed measurement graph indeed takes a star-forest shape; see Table 2 for details of the $G$ matrix. More importantly, the aforementioned five latent classes can be nicely formulated as five different binary configurations of the three latent risks, as (0, 0, 0), (1, 0, 0), (1, 0, 1), (1, 1, 0), and (1, 1, 1), respectively. Here $\alpha_k = 1$ indicates the higher risk group while $\alpha_k = 0$ indicates the lower risk group. See Table 3 for the multidimensional binary configurations of latent classes.

Table 2: Prevention science survey example reformulated using the BLESS model. Latent-to-observed measurement graph structure $G_{6 \times 3}$.

<table>
<thead>
<tr>
<th>Item Content</th>
<th>$\alpha_1$ Household risk</th>
<th>$\alpha_2$ Peer risk</th>
<th>$\alpha_3$ Neighborhood risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Household poverty</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Single-parent status</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Peer cigarette use</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Peer alcohol use</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Neighborhood unemployment</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Neighborhood poverty</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 3: Prevention science survey example reformulated using the BLESS model. Five latent classes obtained and explained in Lanza and Rhoades (2013), and reformulated in the interpretable multidimensional-binary latent variable format.

<table>
<thead>
<tr>
<th>Latent Class Explanation</th>
<th>Fine-grained Latent Risks</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha_1$</td>
<td>$\alpha_2$</td>
<td>$\alpha_3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Household risk</td>
<td>Peer risk</td>
<td>Neighborhood risk</td>
<td></td>
</tr>
<tr>
<td>1 Overall low risk</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2 Peer risk</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3 Household &amp; neighborhood risk</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4 Household &amp; peer risk</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5 Overall high risk</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Because $G$ shows that each latent risk has exactly two observed children characteristics, this example analysed in Lanza and Rhoades (2013) can be exactly regarded as satisfying the minimal conditions for generic identifiability of the BLESS model. As Lanza and Rhoades (2013) did not include the original dataset that they analyzed which is extracted and sampled from the NLSAH survey, we do not perform the test here but point out the testing procedure is just the same as what we conducted in Section 5.1 for the TIMSS data. Specifically, one could simply test the hypothesis of identifiability by testing the marginal independence of the three groups of binary characteristics falling under the household risk, peer risk, and neighborhood risk, respectively. One plausible conjecture is these three risks are likely interdependent due to the interactions of an adolescent’s household, peers, and neighborhood. In such a case, the BLESS model would be identifiable when applied to the survey dataset, and one could use the BLESS model as a more fine-grained and interpretable graphical modeling alternative to plain latent class analysis.

6 Concluding Remarks

This work reveals an interesting and highly nontrivial phenomenon, blessing of latent dependence on identifiability, for the BLESS model, a class of discrete statistical models with multiple binary latent variables. We have proved that under the minimal conditions for generic identifiability that each latent variable has two observed children, the model parameters are identifiable if and only if there exists dependence between the latent variables. Using two real-world examples in education and prevention science, we have shown how our
sharp identifiability results can be applied in practice and guide the use of interpretable, graphical, and more fine-grained latent variable modeling approaches.

The blessing of dependence phenomenon between latent variables is perhaps a bit surprising, partly because the independence assumption of latent variables is predominant in many latent variable modeling approaches. For example, in the traditional and popular factor analysis model, the latent factors are often assumed independent with a diagonal covariance matrix (Anderson and Rubin, 1956). In practice, however, especially in confirmatory latent variable analysis widely seen in education, psychology, and epidemiology, each latent construct of interest carries a substantive meaning (see the examples in Section 5). So it is highly likely that such latent constructs postulated by domain experts are dependent on each other. From this perspective, our theoretical result in this work provides reassurance that the dependence of latent variables can be a blessing, rather than a curse. In the future, it would be interesting to explore whether similar blessing-of-dependence phenomenon is present in other types of graphical latent variable models.

Finally, in a study of the geometry of the simplest discrete latent variable model, the latent class model, and in the special case involving only a total number of $p = 2$ observed variables, Fienberg et al. (2009) made the following remark, “The study of higher dimensional tables is still an open area of research. The mathematical machinery required to handle larger dimensions is considerably more complicated”. Indeed, due to the complex nonlinearity of discrete models with latent structures, previous studies about identifiability either draw on Kruskal’s Theorem or focus on small number of variables (e.g. Allman et al., 2015). In contrast, this work provides a new algebraic technique useful to study the identifiability and geometry of general $p$-dimensional tables. This technique has proved to be more powerful than Kruskal’s theorem when applied to the BLESS model considered in this work, and we are able to use it to derive sharp identifiability results which Kruskal’s Theorem cannot obtain, in addition to revealing the new geometry. Using the new technique to study other properties (beyond identifiability) of discrete graphical latent variable models and exploring its connection to other algebraic statistical techniques would be an interesting future direction.
Supplementary material

The Supplementary Material contains the proofs of all the theoretical results and the details of the EM algorithms.

References


Supplement to “Blessing of Dependence: Identifiability and Geometry of Discrete Models with Multiple Binary Latent Variables”

The Supplementary Material contains the proofs of all the theoretical results and the details of the EM algorithms. Specifically, Section S.1 proves the main theorem on the blessing-of-dependence geometry. Section S.2 presents additional proofs of other theoretical results. Section S.3 presents two EM algorithms.

S.1 Proof of the Main Theorem on the Blessing of Dependence

S.1.1 Proof of Theorem 2

In the following we prove part (a) and part (b) sequentially.

Proof of Part (a) of Theorem 2. First note that under the assumptions of the current theorem, we can apply the previous Theorem 1 to obtain that the matrix $G$ is identifiable. So it remains to consider how to identify $(\theta, \nu)$. Suppose alternative parameters $(\bar{\theta}, \bar{\nu})$ lead to the same distribution of the observables as the true parameters $(\theta, \nu)$.

We first consider an arbitrary index $k \in [K]$ and an arbitrary binary pattern $\alpha' \in \{0, 1\}^K$ with $\alpha'_k = 0$. For this $\alpha'$ and an arbitrary $c \in [d - 1]$ define

$$\Delta_c = \theta_{c|0}^{(k)} e_k + \theta_{c|1}^{(K+k)} e_{K+k} + \sum_{1 \leq m \neq k \leq K} \theta_{c|0}^{(m)} + \sum_{\alpha'_m = 0} \theta_{c|1}^{(m)}, \quad (S.1)$$

$$y_c = c \sum_{j=1}^{2K} e_j.$$

With the above definition, we claim that the row vector of $\bigotimes_{j \in [p]} \left( \Phi^{(j)} - \Delta_{j*} \cdot 1_{2K}^\top \right)$ indexed by response pattern $y_c$ is an all-zero vector. This is true because due to the first two terms
in $\Delta_{1:2K,c}$, any entry in this row must contain a factor of
\[
\left(\frac{\theta^{(k)}_{c|\alpha}}{-\theta^{(k)}_{c|0}}\right)\left(\frac{\theta^{(K+k)}_{c|\alpha}}{-\theta^{(K+k)}_{c|1}}\right),
\]
and this factor must be zero because if $\theta^{(k)}_{c|\alpha} - \theta^{(k)}_{c|0} \neq 0$ then $\alpha \succeq g_k = g_{K+k}$, and then
\[
\theta^{(K+k)}_{c|\alpha} - \theta^{(K+k)}_{c|1} = 0
\]
must hold. Therefore according to (9),
\[
0 = \bigotimes_{j \in [p]} \left(\Phi^{(j)} - \Delta_{j,:} \cdot 1_{2K}^{\top}\right)_{y,c,\alpha} \cdot \nu
\]
\[
= \bigotimes_{j \in [p]} \left(\Phi^{(j)} - \Delta_{j,:} \cdot 1_{2K}^{\top}\right)_{y,c,\alpha} \cdot \nu
\]
\[
= \sum_{\alpha \in \{0,1\}^K} \bigotimes_{j \in [p]} \left(\Phi^{(j)} - \Delta_{j,:} \cdot 1_{2K}^{\top}\right)_{y,c,\alpha} \cdot \nu_{\alpha}
\]
Now due to the third term $\sum_{1 \leq m(\neq k) \leq K} \theta^{(m)}_{c|0}$ in the definition of $\Delta_{1:2K,c}$ in (S.1), the entry $t_{y,c,\alpha} := \bigotimes_{j \in [p]} \left(\Phi^{(j)} - \Delta_{j,:} \cdot 1_{2K}^{\top}\right)_{y,c,\alpha}$ contains a factor
\[
\prod_{1 \leq m(\neq k) \leq K} \left(\theta^{(m)}_{c|\alpha} - \theta^{(m)}_{c|0}\right),
\]
and this factor would equal zero if for some $m \in [K], m \neq k$ there is $\alpha'_m = 1$ but $\alpha_m = 0$. Similarly, due to the fourth term $\sum_{1 \leq m(\neq k) \leq K} \theta^{(m)}_{c|1}$ in the definition of $\Delta_{1:2K,c}$ in (S.1), the entry $t_{y,c,\alpha}$ contains a factor
\[
\prod_{1 \leq m(\neq k) \leq K} \left(\theta^{(m)}_{c|\alpha} - \theta^{(m)}_{c|1}\right),
\]
and this factor would equal zero if for some $m \in [K], m \neq k$ there is $\alpha'_m = 0$ but $\alpha_m = 1$. Summarizing the above two situations, we have that $t_{y,c,\alpha} = 0$ if binary pattern $\alpha$ does not exactly equal pattern $\alpha'$ on all but the $k$th entry. Recall that $\alpha'_k = 0$. Denote by $\alpha' + e_k$ the binary pattern that equals $\alpha'$ on all but the $k$th entry, with the $k$th entry being one. Then
the previously obtained equality $\sum_{\alpha \in \{0,1\}^K} t_{y,c,\alpha} \cdot \nu_\alpha = 0$ can be written as

$$0 = \sum_{\alpha \in \{0,1\}^K} t_{y,c,\alpha} \cdot \nu_\alpha = t_{y,c,\alpha'} \cdot \nu_{\alpha'} + t_{y,c,\alpha'+e_k} \cdot \nu_{\alpha'+e_k}$$

$$= \prod_{1 \leq m(\neq k) \leq K} \left( \theta_{\alpha|c|1}^{(m)} - \theta_{\alpha|c|0}^{(m)} \right) \times \prod_{1 \leq m(\neq k) \leq K} \left( \theta_{\alpha|c|0}^{(m)} - \theta_{\alpha|c|1}^{(m)} \right)$$

$$\times \left\{ \nu_{\alpha'} \left( \theta_{\alpha|c|0}^{(k)} - \overline{\theta}_{\alpha|c|0}^{(k)} \right) \left( \theta_{\alpha|c|1}^{(K+k)} - \overline{\theta}_{\alpha|c|1}^{(K+k)} \right) + \nu_{\alpha'+e_k} \left( \theta_{\alpha'+e_k|c|0}^{(k)} - \overline{\theta}_{\alpha'+e_k|c|0}^{(k)} \right) \left( \theta_{\alpha'+e_k|c|1}^{(K+k)} - \overline{\theta}_{\alpha'+e_k|c|1}^{(K+k)} \right) \right\},$$

where the last equality above follows from two facts (1) $\theta_{\alpha|c|0}^{(k)} = \theta_{\alpha|c|0}^{(k)}$ due to $\alpha_k' = 0$; and (2) $\theta_{\alpha|c|1}^{(K+k)} = \theta_{\alpha|c|1}^{(K+k)}$ due to $(\alpha' + e_k)_k = 1$. Now note that in the above display, the first two product factors are nonzero because of the assumption (1), we obtain

$$\nu_{\alpha'} \left( \theta_{\alpha|c|0}^{(k)} - \overline{\theta}_{\alpha|c|0}^{(k)} \right) \left( \theta_{\alpha|c|1}^{(K+k)} - \overline{\theta}_{\alpha|c|1}^{(K+k)} \right) + \nu_{\alpha'+e_k} \left( \theta_{\alpha'+e_k|c|0}^{(k)} - \overline{\theta}_{\alpha'+e_k|c|0}^{(k)} \right) \left( \theta_{\alpha'+e_k|c|1}^{(K+k)} - \overline{\theta}_{\alpha'+e_k|c|1}^{(K+k)} \right) = 0. \quad (S.2)$$

Note that the above key equation holds for an arbitrary $\alpha'$ with $\alpha_k' = 0$ and also for an arbitrary $c \in \{1, \ldots, d - 1\}$. For each $c$ define

$$x_{0k,c} = \left( \theta_{\alpha|c|0}^{(k)} - \overline{\theta}_{\alpha|c|0}^{(k)} \right) \left( \theta_{\alpha|c|1}^{(K+k)} - \overline{\theta}_{\alpha|c|1}^{(K+k)} \right), \quad (S.3)$$

$$x_{1k,c} = \left( \theta_{\alpha|c|1}^{(k)} - \overline{\theta}_{\alpha|c|1}^{(k)} \right) \left( \theta_{\alpha|c|1}^{(K+k)} - \overline{\theta}_{\alpha|c|1}^{(K+k)} \right),$$

and note that only the first factor of $x_{0k,c}$ and only the second factor of $x_{1k,c}$ can potentially be zero. Then we have that

$$x_{0k,c} = 0 \text{ if and only if } \theta_{\alpha|c|0}^{(k)} = \overline{\theta}_{\alpha|c|0}^{(k)}; \quad (S.4)$$

$$x_{1k,c} = 0 \text{ if and only if } \theta_{\alpha|c|1}^{(K+k)} = \overline{\theta}_{\alpha|c|1}^{(K+k)}. \quad (S.5)$$

Then with $\alpha'$ ranging over all the $2^{K-1}$ possible configurations and $c$ ranging over $\{1, \ldots, d - 1\}$...
Eq. (S.2) implies the following systems of equations hold,

\[
\begin{pmatrix}
\nu_{\alpha'}^{(1)} & \nu_{\alpha'}^{(1)} + e_k \\
\nu_{\alpha'}^{(2)} & \nu_{\alpha'}^{(2)} + e_k \\
\vdots & \\
\nu_{\alpha'}^{(2K-1)} & \nu_{\alpha'}^{(2K-1)} + e_k
\end{pmatrix}
\begin{pmatrix}
x_{0,k,1} & x_{0,k,2} & \cdots & x_{0,k,d-1} \\
x_{1,k,1} & x_{1,k,2} & \cdots & x_{1,k,d-1}
\end{pmatrix}
= 0_{2^{K-1} \times (d-1)},
\]

where \(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(2K-1)}\) represent the \(\{0, 1\}^{K-1} = 2^{K-1}\) possible configurations \(\alpha'\) can take, all having the \(k\)th entry being zero. We denote the \(2^{K-1} \times 2\) matrix on the left hand side of (S.6) consisting of \(\nu\)'s by \(P^{(k)}\), and denote its first column by \(P^{(k)}_{:1}\) and its second column by \(P^{(k)}_{:2}\). The system (S.6) can be written as

\[
x_{0,k,c}P^{(k)}_{:1} + x_{1,k,c}P^{(k)}_{:2} = 0_{2^{K-1} \times 1}, \quad c = 1, \ldots, d - 1,
\]

therefore we know that \((x_{0,k,1}, x_{1,k,1}) = \cdots = (x_{0,k,d-1}, x_{1,k,d-1}) = (0, 0)\) holds if vectors \(P^{(k)}_{:1}\) and \(P^{(k)}_{:2}\) are linearly independent. The values of \(\nu = (\nu_{\alpha})\) that would yield the two vectors \(P^{(k)}_{:1}\) and \(P^{(k)}_{:2}\) linearly dependent are the zero set of several polynomial equations with \(\nu\)'s as indeterminates; that is,

\[
N_k = \{\nu : P^{(k)}_{:1} \text{ and } P^{(k)}_{:2} \text{ are linearly dependent for } P^{(k)} \text{ defined in (S.6)}.\}
\]  

is a zero set of several nonzero polynomials in \(\nu\)'s. Hence \(N_k\) forms an algebraic subvariety of the parameter space \(\Delta^{2^{K-1}}\) of \(\nu\) and \(N_k\) has measure zero with respect to the Lebesgue measure on \(\Delta^{2^{K-1}}\). Further, recall that as long as \(\nu \notin N_k\) and \(\nu_{\alpha} > 0\), there is \(x_{0,k,c} = x_{1,k,c} = 0\) which implies the identifiability of \(\theta_{c|0}^k\) and \(\theta_{c|1}^{K+k}\) as shown in (S.4) and (S.5). Summarizing the conclusion for all the \(k \in [K]\), we have that as long as

\[
\nu \notin \bigcup_{k \in [K]} N_k,
\]

all the \(\theta\)-parameters will be identifiable. Since \(\bigcup_{k \in [K]} N_k \subseteq \Delta^{2^{K-1}}\) has measure zero with respect to the Lebesgue measure on \(\Delta^{2^{K-1}}\), we have essentially shown that \(\theta\) are generically
identifiable. Further, when \( \boldsymbol{\nu} \not\in \bigcup_{k \in [K]} \mathcal{N}_k \) and \( \boldsymbol{\theta} \) are identifiable with \( \Phi_j = \overline{\Phi}_j \) for all \( j \), we next show that \( \boldsymbol{\nu} \) are also identifiable. Consider the equations given by the first \( K \) observed variables,

\[
\left( \bigotimes_{j \in [K]} \Phi^{(j)} \right) \cdot \boldsymbol{\nu} = \left( \bigotimes_{j \in [K]} \overline{\Phi}^{(j)} \right) \cdot \overline{\boldsymbol{\nu}},
\]

Given an arbitrary binary pattern \( \alpha \) and any \( c \in \{1, \ldots, d-1\} \), define

\[
\Delta_{\alpha,c} = \sum_{1 \leq k \leq K} \theta^{(k)}_{c|0} e_k + \sum_{1 \leq k \leq K} \theta^{(k)}_{c|1} e_k, \quad y_c = c \sum_{k=1}^{K} e_k.
\]

For any \( \alpha' \in \{0, 1\}^K \), denote by \( t_{y_c, \alpha'} \) and \( \overline{t}_{y_c, \alpha'} \) the element in \( \bigotimes_{j \in [K]} \Phi^{(j)} \) and \( \bigotimes_{j \in [K]} \overline{\Phi}^{(j)} \), respectively, indexed by response pattern \( y_c \) and latent pattern \( \alpha' \). Then Lemma 1 and (9) indicate that \( t_{y_c, \alpha'} \neq 0 \) only if \( \alpha' = \alpha \) and \( \overline{t}_{y_c, \alpha'} \neq 0 \) only if \( \alpha' = \alpha \). Therefore

\[
\prod_{1 \leq k \leq K} \left( \theta^{(k)}_{c|1} - \theta^{(k)}_{c|0} \right) \prod_{1 \leq k \leq K} \left( \theta^{(k)}_{c|0} - \theta^{(k)}_{c|1} \right) \nu_{\alpha} = \prod_{1 \leq k \leq K} \left( \theta^{(k)}_{c|1} - \theta^{(k)}_{c|0} \right) \prod_{1 \leq k \leq K} \left( \theta^{(k)}_{c|0} - \theta^{(k)}_{c|1} \right) \nu_{\alpha},
\]

which further gives \( \nu_{\alpha} = \nu_{\alpha} \) because \( \prod_{1 \leq k \leq K} \left( \theta^{(k)}_{c|1} - \theta^{(k)}_{c|0} \right) \prod_{1 \leq k \leq K} \left( \theta^{(k)}_{c|0} - \theta^{(k)}_{c|1} \right) \neq 0 \) under the assumption (1). Since \( \alpha \) above is arbitrary, we have obtained \( \nu = \nu \). Thus far we have proven that as long as \( \nu \) satisfies (S.8), there must be \( (\theta, \nu) = (\overline{\theta}, \overline{\nu}) \). This establishes the generic identifiability of all the model parameters and completes the proof of part (a) of the theorem.

**Proof of Part (b) of Theorem 2.** We next prove part (b) of the theorem by showing that the two vectors \( P_{i,1}^{(k)} \) and \( P_{i,2}^{(k)} \) are linearly dependent if and only if \( \alpha_k \) and other latent variables are statistically independent. If the two vectors \( P_{i,1}^{(k)} \) and \( P_{i,2}^{(k)} \) are linearly dependent, then with out loss of generality we can assume \( P_{i,2}^{(k)} = \rho \cdot P_{i,1}^{(k)} \) for some \( \rho \neq 0 \). Then
by (S.6), there is \( \nu_{\alpha'(\ell)+e_k} = \rho \cdot \nu_{\alpha'(\ell)} \) for \( \ell = 1, \ldots, 2^{K-1} \), which implies

\[
\nu_{\alpha'(\ell)+e_k} \cdot \nu_{\alpha'(m)+e_k} = \nu_{\alpha'(\ell)} \cdot \nu_{\alpha'(m)}, \quad \text{for any } 1 \leq m, \ell \leq 2^{K-1}.
\]

(S.9)

Denote \((\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_K) =: \alpha_{-k}\). Since all the \(\alpha_m\)'s are binary, for any \(s \in \{0, 1\}^{K-1}\) and any \(t \in \{0, 1\}\) we have

\[
\mathbb{P}(\alpha_{-k} = s, \alpha_k = 1) = \mathbb{P}(\alpha_{-k} = s, \alpha_k = 1) \left( \sum_{r \in \{0, 1\}^K} \nu_r \right)
\]

\[
= \mathbb{P}(\alpha_{-k} = s, \alpha_k = 1) \left\{ \sum_{\alpha' \in \{0, 1\}^K \atop \alpha'_{k} = 0} \left( \nu_{\alpha' + e_k} + \nu_{\alpha'} \right) \right\}
\]

\[
= \sum_{\alpha' \in \{0, 1\}^K \atop \alpha'_{k} = 0} \left\{ \mathbb{P}(\alpha_{-k} = s, \alpha_k = 1) \cdot \nu_{\alpha' + e_k} + \mathbb{P}(\alpha_{-k} = s, \alpha_k = 1) \cdot \nu_{\alpha'} \right\}
\]

(S.9)

\[
= \sum_{\alpha' \in \{0, 1\}^K \atop \alpha'_{k} = 0} \left\{ \mathbb{P}(\alpha_{-k} = s, \alpha_k = 1) \cdot \nu_{\alpha' + e_k} + \mathbb{P}(\alpha_{-k} = s, \alpha_k = 0) \cdot \nu_{\alpha' + e_k} \right\}
\]

\[
= \sum_{\alpha' \in \{0, 1\}^K \atop \alpha'_{k} = 0} \mathbb{P}(\alpha_{-k} = s) \nu_{\alpha' + e_k}
\]

\[
= \mathbb{P}(\alpha_{-k} = s) \cdot \mathbb{P}(\alpha_k = 1),
\]

(S.10)

where the last but third equality results from \(\mathbb{P}(\alpha_{-k} = s, \alpha_k = 1) \cdot \nu_{\alpha'} = \mathbb{P}(\alpha_{-k} = s, \alpha_k = 0) \cdot \nu_{\alpha' + e_k} \) by (S.9). Further, we can show that

\[
\mathbb{P}(\alpha_{-k} = s, \alpha_k = 0) = \mathbb{P}(\alpha_{-k} = s) - \mathbb{P}(\alpha_{-k} = s, \alpha_k = 1)
\]

\[
= \mathbb{P}(\alpha_{-k} = s) - \mathbb{P}(\alpha_{-k} = s) \cdot \mathbb{P}(\alpha_k = 1)
\]

\[
= \mathbb{P}(\alpha_{-k} = s) \cdot \mathbb{P}(\alpha_k = 0).
\]

(S.11)

The above two conclusions (S.10) and (S.11) indicate \(\alpha_k\) and \(\alpha_{-k}\) are statistically independent, that is, \(\alpha_k \perp \perp \alpha_{-k}\). Recall the definition in (S.7) that \(\mathcal{N}_k = \{\nu : \mathbf{F}^{(k)}_{i,1} \text{ and } \mathbf{P}^{(k)}_{i,1} \text{ defined in (S.7) are linearly dependent.}\} \), and now we have shown
(C1) \( \nu \not\in N_k \implies \theta_i^{(k)} \) and \( \theta_i^{(K+k)} \) are identifiable.

(C2) \( \nu \in N_k \implies \alpha_k \perp \perp \alpha_{-k} \).

Next we show that the reverse directions of the above two claims (C1) and (C2) also hold; namely, we next show that

(\widehat{C1}) If \( \theta_i^{(k)} \) and \( \theta_i^{(K+k)} \) are identifiable, then \( \nu \not\in N_k \);

(\widehat{C2}) If \( \alpha_k \perp \perp \alpha_{-k} \), then \( \nu \in N_k \).

Suppose \( \alpha_k \perp \perp \alpha_{-k} \), then for any \( s \in \{0, 1\}^{K-1} \) and \( z \in \{0, 1\} \) there is \( \mathbb{P}(\alpha_{-k} = s, \alpha_k = z) = \mathbb{P}(\alpha_{-k} = s) \cdot \mathbb{P}(\alpha_k = z) \), which implies the following,

for any \( s \in \{0, 1\}^{K-1} \),

\[
\begin{cases}
    \frac{\mathbb{P}(\alpha_{-k} = s, \alpha_k = 1)}{\mathbb{P}(\alpha_{-k} = s)} = \mathbb{P}(\alpha_k = 1) := \rho_1;
    \\
    \frac{\mathbb{P}(\alpha_{-k} = s, \alpha_k = 0)}{\mathbb{P}(\alpha_{-k} = s)} = \mathbb{P}(\alpha_k = 0) := \rho_0.
\end{cases}
\]

Taking the ratio of the above two equalities gives

\[
\frac{\mathbb{P}(\alpha_{-k} = s, \alpha_k = 1)}{\mathbb{P}(\alpha_{-k} = s, \alpha_k = 0)} = \frac{\rho_1}{\rho_0} \text{ for any } s \in \{0, 1\}^{K-1}.
\]

Recalling the definition of the \( 2^{K-1} \times 2 \) matrix \( P^{(k)} \) in (S.6), the above equality exactly means the two vectors \( P_{i,1}^{(k)} \) and \( P_{i,2}^{(k)} \) are linearly dependent. So we have shown (\widehat{C2}) holds.

Finally, to show (\widehat{C1}) holds, it suffices to prove that if \( \nu \in N_k \), then \( \theta_i^{(k)} \) and \( \theta_i^{(K+k)} \) are not identifiable. To this end, we next explicitly construct alternative parameters \( \overline{\theta}_i^{(k)} \) and \( \overline{\theta}_i^{(K+k)} \) that lead to the same distributions of the observables as the true parameters \( \theta_i^{(k)} \) and \( \theta_i^{(K+k)} \). If \( \nu \in N_k \), then without loss of generality we can assume there exists \( \rho > 0 \) such
that \( \nu_{\alpha'+e_k} = \rho \cdot \nu_{\alpha'} \) for any \( \alpha' \) with \( \alpha'_k = 0 \). Then equations (S.2) now become

\[
\left( \theta_{c|0}^{(k)} - \overline{\theta}_{c|0}^{(k)} \right) \left( \theta_{c|0}^{(K+k)} - \overline{\theta}_{c|1}^{(K+k)} \right) + \rho \cdot \left( \theta_{c|1}^{(k)} - \overline{\theta}_{c|0}^{(k)} \right) \left( \theta_{c|1}^{(K+k)} - \overline{\theta}_{c|1}^{(K+k)} \right) = 0.
\]

Now considering an arbitrary \( \overline{\theta}_{c|0}^{(k)} \) in a small neighborhood of the true parameter \( \theta_{c|0}^{(k)} \), we can solve for \( \overline{\theta}_{c|1}^{(K+k)} \) as

\[
\overline{\theta}_{c|1}^{(K+k)} = \theta_{c|1}^{(K+k)} - \frac{\left( \theta_{c|0}^{(k)} - \overline{\theta}_{c|0}^{(k)} \right) \left( \theta_{c|0}^{(K+k)} - \overline{\theta}_{c|1}^{(K+k)} \right)}{\left( \theta_{c|0}^{(k)} - \overline{\theta}_{c|0}^{(k)} \right) + \rho \cdot \left( \theta_{c|1}^{(k)} - \overline{\theta}_{c|0}^{(k)} \right)}.
\]

Note that the above \( \overline{\theta}_{c|1}^{(K+k)} \) and \( \overline{\theta}_{c|0}^{(k)} \) satisfy all the equations in

\[
\left( \bigcirc_{j \in [p]} \Phi^{(j)} \right) \cdot \nu = \left( \bigcirc_{j \in [p]} \overline{\Phi}^{(j)} \right).
\]

\( \nu \). We have thus shown that \( \theta_{c|1}^{(K+k)} \) and \( \theta_{c|0}^{(k)} \) are not identifiable and prove the previous claim (\( \tilde{\text{C}1} \)).

In summary, now that we have proven (C1), (\( \tilde{\text{C}1} \)), (C2), (\( \tilde{\text{C}2} \)), there are

\[
\theta_{z}^{(k)} \text{ and } \theta_{z}^{(K+k)} \text{ are identifiable.}
\]

\( \iff \) \( \nu \not\in N_k = \{ \nu : P_{i|1}^{(k)} \text{ and } P_{i|1}^{(k)} \text{ are linearly dependent.} \} \)

\( \iff \) \( \alpha_k \perp \perp \alpha_{-k} \).

This completes the proof of Theorem 2. \( \square \)

### S.2 Additional Proofs

#### S.2.1 Proof of Proposition 1

Under the condition of the proposition, we construct a non-identifiable example as follows. Without loss of generality, suppose the first latent variable has only one child and the binary matrix \( G \) takes the following form

\[
G = \begin{pmatrix}
1 & 0 \\
0 & G^*
\end{pmatrix},
\]

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where $G^*$ has size $(p - 1) \times (K - 1)$. Given arbitrary valid model parameters $(\nu, \theta)$, we next construct an alternative set of parameters $(\nu', \theta') \neq (\nu, \theta)$ such that $(\nu', \theta')$ and $(\nu, \theta)$ lead to the same distribution of the observed response vector $r$. Suppose $\theta^{(j)}_{c_j|x} = \theta^{(j)}_{c_j|x}$ for all $j \in \{2, \ldots, p\}$, $c_j \in [d]$, and $x \in \{0, 1\}$. Then $P_\nu(r \mid \nu, \theta) = P_\nu(r \mid \nu', \theta')$ implies the following equations

$$\forall \alpha^* \in \{0, 1\}^{K - 1}, \quad \forall c \in [d], \quad \theta^{(1)}_{c|0} \nu_{0, \alpha^*} + \theta^{(1)}_{c|1} \nu_{1, \alpha^*} = \bar{\theta}^{(1)}_{c|0} \nu_{0, \alpha^*} + \bar{\theta}^{(1)}_{c|1} \nu_{1, \alpha^*}.$$

For each possible $\alpha^* \in \{0, 1\}^{K - 1}$, we sum the $d$ equations above for $c = 1, \ldots, d$ and further obtain $\nu_{0, \alpha^*} + \nu_{1, \alpha^*} = \nu_{0, \alpha^*} + \nu_{1, \alpha^*}$. Therefore the above system of equations are equivalent to the following,

$$\forall \alpha^* \in \{0, 1\}^{K - 1}, \quad \begin{cases} 
\nu_{0, \alpha^*} + \nu_{1, \alpha^*} = \nu_{0, \alpha^*} + \nu_{1, \alpha^*}; \\
\theta^{(1)}_{c|0} \nu_{0, \alpha^*} + \theta^{(1)}_{c|1} \nu_{1, \alpha^*} = \bar{\theta}^{(1)}_{c|0} \nu_{0, \alpha^*} + \bar{\theta}^{(1)}_{c|1} \nu_{1, \alpha^*}, \quad c \in [d].
\end{cases}$$

We next set $\bar{\theta}^{(1)}_{c|0} = \theta^{(1)}_{c|0}$ for all $c \in \{1, \ldots, d\}$, and take the alternative $\bar{\theta}^{(1)}_{c|1}$ from an arbitrarily small neighborhood of the true parameter $\theta^{(1)}_{1|1}$ with $\bar{\theta}^{(1)}_{1|1} \neq \theta^{(1)}_{1|1}$. Then

$$\begin{cases} 
\bar{\nu}_{1, \alpha^*} = \nu_{1, \alpha^*} \cdot \frac{\theta^{(1)}_{1|1} - \theta^{(1)}_{1|0}}{\bar{\theta}^{(1)}_{1|1} - \theta^{(1)}_{1|0}}, \quad \forall \alpha^* \in \{0, 1\}^{K - 1}; \\
\bar{\nu}_{0, \alpha^*} = \nu_{0, \alpha^*} + \nu_{1, \alpha^*} \cdot \frac{\theta^{(1)}_{1|1} - \theta^{(1)}_{1|0}}{\bar{\theta}^{(1)}_{1|1} - \theta^{(1)}_{1|0}}, \quad \forall \alpha^* \in \{0, 1\}^{K - 1}; \\
\bar{\theta}^{(1)}_{c|1} = \theta^{(1)}_{c|0} + (\theta^{(1)}_{c|1} - \theta^{(1)}_{c|0}) \cdot \frac{\theta^{(1)}_{1|1} - \theta^{(1)}_{1|0}}{\bar{\theta}^{(1)}_{1|1} - \theta^{(1)}_{1|0}}, \quad \forall c = 2, \ldots, d;
\end{cases}$$

Since $\bar{\theta}^{(1)}_{1|1} \neq \theta^{(1)}_{1|1}$, one can see that generally there is $\bar{\nu}_{1, \alpha^*} \neq \nu_{1, \alpha^*}$, $\bar{\nu}_{0, \alpha^*} \neq \nu_{0, \alpha^*}$, and $\bar{\theta}^{(1)}_{c|1} \neq \theta^{(1)}_{c|1}$ for $c = 2, \ldots, d$ based on the above equations. Note that the alternative parameter $\bar{\theta}^{(1)}_{1|1}$ can be chosen from an arbitrarily small neighborhood of the true parameter $\theta^{(1)}_{1|1}$, so we have proven that even local identifiability fails to hold in the considered setting. This completes the proof of Proposition 1. \hspace{1cm} \Box
We introduce the following useful lemma.

**Lemma 2.** Consider true graphical matrix $G$ and associated true parameters $\theta, \nu$ that satisfy (1), suppose alternative $\overline{G}, \overline{\theta}, \overline{\nu}$ lead to the same distribution of the observed vector $y$ as the true parameters. Then $\overline{\theta}$ and $\theta$ must satisfy

$$\overline{\theta}^{(j)}_{c|0} \neq \theta^{(j)}_{c|1}, \quad \overline{\theta}^{(j)}_{c|1} \neq \theta^{(j)}_{c|0}$$

for any $c \in [d]$.

**S.2.2 Proof of Theorem 1**

We prove the theorem in two steps.

**Step 1.** In this step we prove the following lemma.

**Lemma 3.** Suppose $G = (I_K, I_K)^\top$, which vertically stacks two identity submatrices $I_K$. Consider that $(G, \theta, \nu)$ and $(\overline{G}, \overline{\theta}, \overline{\nu})$ lead to the same distribution of the observed vector $y$. For an arbitrary $h \in [K]$, if there exists two sets $A \subseteq [K] \setminus \{h\}$ and $B \subseteq \{K + 1, \ldots, J\}$ such that $G$ satisfies

$$\max_{m \in B} g_{m,h} = 0,$$

$$\max_{m \in B} g_{m,k} = 1 \text{ for all } k \in A,$$

then $\overline{G}$ must satisfy $\bigvee_{k \in A} \overline{g}_k \not\geq \overline{g}_h$.

Please see the proof of Lemma 3 in the Supplementary Material.

**Step 2.** In this step we show $\overline{G} = (I_K, I_K)^\top$ holds up to a column permutation. Let $B = \{K + 1, \ldots, 2K\}$ and $A_h = [K] \setminus \{h\}$ for an arbitrary index $h \in [K]$. Then the condition in Lemma 3 is satisfied and

$$\bigvee_{k \in A_h} g_k \not\geq \overline{g}_h,$$

which implies that the row vector $\overline{g}_h$ contains an entry of “1” in some column $q_h$ with all the $\overline{g}_k$ in $A_h$ having “0” in this column $q_h$. Since the above holds for all the $h \in [K]$, we obtain that the $K$ row vectors $\overline{g}_1, \ldots, \overline{g}_K$ contains “1”s in $K$ different columns. This exactly
implies that $\bar{G}_{1:K}$ equals the identity matrix $I_K$ up to a column permutation. Since the first $K$ rows and the second $K$ rows of $G$ are both $I_K$, by symmetry to the above deduction we can also obtain that $\bar{G}_{(K+1):2K}$ equals $I_K$ up to a column permutation.

Now it only remains to show that the column permutations of $\bar{G}_{1:K}$ and $\bar{G}_{(K+1):2K}$ are the same. Suppose $\bar{g}_k = \bar{g}_{K+k'}$ for some $k, k' \in [K]$. Define

$$\Delta_{1:p,c} = \theta^{(k)}_{c|0} e_k + \theta^{(K+k')}_{c|1} e_{K+k'},$$

$$y_c = c(e_k + e_{K+k'}).$$

With this definition, we claim that the row vector corresponding to response pattern $y_c$ of $\bigodot_{j \in [p]} \left( \Phi^{(j)} - \Delta_j \cdot 1_{2K}^T \right)$ must be a zero-vector. This is because any entry in this row must contain a factor of

$$\left( \theta^{(k)}_{c|\alpha} - \theta^{(k)}_{c|0} \right) \left( \theta^{(K+k')}_{c|\alpha} - \theta^{(K+k')}_{c|1} \right),$$

and this factor must be zero because if $\theta^{(k)}_{c|\alpha} - \theta^{(k)}_{c|0} \neq 0$ then $\alpha \succeq g_k = g_{K+k'}$, and then $\theta^{(K+k')}_{c|\alpha} - \theta^{(K+k')}_{c|1} = 0$ must hold. Now that $\bigodot_{j \in [p]} \left( \Phi^{(j)} - \Delta_j \cdot 1_{2K}^T \right) y_c$ is a zero-vector, (9) gives that

$$0 = \bigodot_{j \in [p]} \left( \Phi^{(j)} - \Delta_j \cdot 1_{2K}^T \right) y_c : \nu$$

$$= \bigodot_{j \in [p]} \left( \Phi^{(j)} - \Delta_j \cdot 1_{2K}^T \right) y_c : \nu$$

$$= \left( \theta^{(k)}_{c|1} - \theta^{(k)}_{c|0} \right) \left( \theta^{(K+k')}_{c|0} - \theta^{(K+k')}_{c|1} \right) \left( \sum_{\alpha \supseteq g_k, \alpha \not\supseteq g_{K+k'}} \nu_{\alpha} \right).$$

If the set $\mathcal{M} := \{ \alpha \in \{0,1\}^K : \alpha \supseteq g_k, \alpha \not\supseteq g_{K+k'} \}$ is nonempty, then the above equation gives a contradiction. This means $\mathcal{M}$ must be an empty set, which implies that $g_{K+k'} = g_k$ must hold. Considering the true $G = (I_K, I_K)^\top$, we have that $k' = k$ must hold. Now we have shown that as long as $\bar{g}_k = \bar{g}_{K+k'}$, there is $k = k'$. This essentially shows $\bar{G}_{1:K} = \bar{G}_{(K+1):2K}$ holds. This completes the proof of Theorem 1. 

\qed
S.2.3 Proof of Corollary 1

The conclusion of the corollary has actually been proved in the last part of the proof of Theorem 2(b). To see this, recall that with \( G = (I_K, I_K)^\top \), there is \( \text{Child}(\alpha_k) = \{y_k, y_{K+k}\} \).

Since in the end of the proof of Theorem 2(b) we established that conditional probabilities \( \theta_{i}^{(k)} \) and \( \theta_{i}^{(K+k)} \) are identifiable if and only if \( \alpha_k \not\perp \perp \alpha_{-k} \), this means the parameters associated with \( \text{Child}(\alpha_k) \mid \alpha_k \) are identifiable if and only if \( \alpha_k \not\perp \perp \alpha_{-k} \). This proves Corollary 1.

S.2.4 Proof of Proposition 2

Under the assumption that each latent variable has three children, we show identifiability in a similar fashion as the proof of Theorem 4 in Allman et al. (2009) by using Kruskal’s theorem.

Under the assumption that each latent variable has at least three children variables, suppose without loss of generality that \( G = (I_K, I_K, I_K, G^\star) \), where the submatrix \( G^\star \) can take an arbitrary form. Suppose the alternative parameters \( \theta, \bar{\theta} \) associated with a potentially different \( G \) lead to the same distribution of the \( p \) observed variables. Group the first \( K \) observed variables \( y_1, \ldots, y_K \) into one discrete variable with \( d^K \) categories and denote it by \( z_1 \), then each of the \( d^K \) possible configurations of the vector \( \tilde{c} = (y_1, \ldots, y_K) \) corresponds to one category that \( z_1 \) can take. Similarly group \( y_{K+1}, \ldots, y_{2K} \) into another variable \( z_2 \), and group \( y_{2K+1}, \ldots, y_{3K} \) into another variable \( z_3 \). Then given latent pattern \( \alpha \), the conditional probability table of \( z_1, z_2, z_3 \) each has size \( d^K \times 2^K \); denote such a table by \( \Psi_m \). Based on the star-forest dependence graph structure it is not hard to deduct that each such \( d^K \times 2^K \) table can be written as

\[
\Psi_1 = \bigotimes_{j=1}^{K} \begin{pmatrix} \theta_{1|0} & \theta_{1|1} \\ \vdots & \vdots \\ \theta_{d|0} & \theta_{d|1} \end{pmatrix}, \quad \Psi_2 = \bigotimes_{j=K+1}^{2K} \begin{pmatrix} \theta_{1|0} & \theta_{1|1} \\ \vdots & \vdots \\ \theta_{d|0} & \theta_{d|1} \end{pmatrix}, \quad \Psi_3 = \bigotimes_{j=2K+1}^{3K} \begin{pmatrix} \theta_{1|0} & \theta_{1|1} \\ \vdots & \vdots \\ \theta_{d|0} & \theta_{d|1} \end{pmatrix}.
\]

Recall the assumption 1 that \( \theta_{c|1}^{(j)} > \theta_{c|0}^{(j)} \) for all \( j \in [p] \) and \( c \in [d-1] \), which implies \( \theta_{d|1}^{(j)} < \theta_{d|0}^{(j)} \). Therefore the following inequality always holds for any \( c \in [d-1] \),

\[
\theta_{c|0}^{(j)} \cdot \theta_{d|1}^{(j)} - \theta_{c|1}^{(j)} \cdot \theta_{d|0}^{(j)} < 0,
\]
which implies each \( d \times 2 \) factor matrix in the definition of \( \Psi_1, \Psi_2, \) and \( \Psi_3 \) has full column rank 2. Since the Kronecker product of full-rank matrices is still full-rank, we obtain that each of \( \Psi_1, \Psi_2, \Psi_3 \) has full column rank \( 2^K \).

Next further group the variable \( z_3 \) and all the remaining variables \( y_{3K+1}, \ldots, y_p \) (if they exist) into another discrete variable \( z_4 \) with \( d^{p-2K} \) categories. Denote the conditional probability table of \( z_4 \) by \( \Psi_4 \), which has size \( d^{p-2K} \times 2^K \). Then by definition there is

\[
\Psi_4 = \Psi_3 \odot \Phi_{3K+1} \odot \Phi_{3K+2} \cdots \odot \Phi_p.
\]

Since every matrix in the above Khatri-Rao product is a conditional probability table with each column summing to one, the \( \Psi_3 \) can be obtained by summing appropriate rows of \( \Psi_4 \). This indeed indicates that the column rank of \( \Psi_4 \) will not be smaller than that of \( \Psi_3 \), so \( \Psi_4 \) also has full rank \( 2^K \). Note that for alternative parameters \( \Phi_j \) there is

\[
(\Psi_1 \odot \Psi_2 \odot \Psi_4) \cdot \nu = (\overline{\Psi}_1 \odot \overline{\Psi}_2 \odot \overline{\Psi}_4) \cdot \overline{\nu}
\]

Now we invoke Kruskal’s theorem (Kruskal, 1977) as follows on the uniqueness of three-way tensor decompositions. Let \( M_1, M_2, M_3 \) be three matrices of size \( a_m \times r \) for \( m = 1, 2, 3 \), and \( N_1, N_2, N_3 \) be three matrices each with \( r \) columns. Suppose \( \odot \sum_{m=1}^{3} M_m \cdot 1 = \odot \sum_{m=1}^{3} N_m \cdot 1 \). Denote by \( \text{rank}_{Kr}(M) \) the Kruskal rank of a matrix \( M \), which is the maximum number \( R \) such that every \( R \) columns of \( M \) are linearly independent. If \( \text{rank}_{Kr}(M_1) + \text{rank}_{Kr}(M_2) + \text{rank}_{Kr}(M_3) \geq 2r + 2 \), then Kruskal’s theorem guarantees that there exists a permutation matrix \( P \) and three invertible diagonal matrices \( D_m \) with \( D_1 D_2 D_3 = I_r \) and \( N_m = M_m D_m P \) for each \( m = 1, 2, 3 \).

Based on Kruskal’s theorem stated above, we can show that \( \overline{\Psi}_m = \overline{\Psi}_m \) for \( m = 1, 2, 4 \) and \( \nu = \overline{\nu} \) up to a column latent class permutation. Finally, note that both individual entries \( \overline{\theta}_{c|1}^{(j)}, \overline{\theta}_{c|0}^{(j)} \) and the graphical matrix \( \overline{G} \) can read off from the \( \overline{\Psi}_m \). This implies the \( \overline{\theta} \) and \( \overline{G} \) must also equal the \( \theta \) and \( G \) up to a latent variable permutation. The proof is complete.

**Proof of Proposition 3.** Denote the marginal probability mass function of the vector \((\alpha_{k_1}, \alpha_{k_2})\) by \( \{\vec{v}(\alpha_{k_1}, \alpha_{k_2}); (\alpha_{k_1}, \alpha_{k_2}) \in \{0, 1\}^2\} \). Each \( \vec{v}(\alpha_{k_1}, \alpha_{k_2}) = \mathbb{P}(a_{i,k_1} = \alpha_{k_1}, a_{i,k_2} = \alpha_{k_2}) \) can be ob-
tained by summing up appropriate entries of the vector \( \nu_{\alpha}; \alpha \in \{0,1\}^K \). Similarly, denote
the marginal distribution of each \( \alpha_k \in \{0,1\} \) by \( \tilde{\nu}_{\alpha_k} = \mathbb{P}(a_{i,k} = \alpha_k) \). Then we have

\[
\mathbb{P}(\{ y_j = c_j : j \in \text{Child}(\alpha_{k_1}) \}, \{ y_m = c_m : m \in \text{Child}(\alpha_{k_2}) \}) \\
= \sum_{\alpha \in \{0,1\}^K} \nu_{\alpha} \prod_{j \in \text{Child}(\alpha_{k_1})} \prod_{k=1}^K \left[ \left( \theta^{(j)}_{c_j[1]} \right)^{\alpha_k} \cdot \left( \theta^{(j)}_{c_j[0]} \right)^{1-\alpha_k} \right]^{1(g_{j,k} = 1)} \\
= \sum_{\alpha \in \{0,1\}^K} \nu_{\alpha} \prod_{j \in \text{Child}(\alpha_{k_1})} \mathbb{P}(y_j \mid g_j, \alpha) \\
= \sum_{(\alpha_{k_1}, \alpha_{k_2}) \in \{0,1\}^2} \tilde{\nu}_{(\alpha_{k_1}, \alpha_{k_2})} \prod_{j \in \text{Child}(\alpha_{k_1})} \mathbb{P}(y_j \mid \alpha_{k_1}) \prod_{j \in \text{Child}(\alpha_{k_2})} \mathbb{P}(y_j \mid \alpha_{k_2}) \\
= \left( \sum_{\alpha_{k_1} \in \{0,1\}} \tilde{\nu}_{\alpha_{k_1}} \prod_{j \in \text{Child}(\alpha_{k_1})} \mathbb{P}(y_j \mid \alpha_{k_1}) \right) \cdot \left( \sum_{\alpha_{k_2} \in \{0,1\}} \tilde{\nu}_{\alpha_{k_2}} \prod_{j \in \text{Child}(\alpha_{k_2})} \mathbb{P}(y_j \mid \alpha_{k_2}) \right) \\
= \left( \sum_{\alpha \in \{0,1\}^K} \tilde{\nu}_{\alpha} \prod_{j \in \text{Child}(\alpha)} \mathbb{P}(y_j \mid \alpha, g_j) \right) \cdot \left( \sum_{\alpha \in \{0,1\}^K} \tilde{\nu}_{\alpha} \prod_{j \in \text{Child}(\alpha)} \mathbb{P}(y_j \mid \alpha, g_j) \right) \\
= \mathbb{P}(\{ y_j = c_j : j \in \text{Child}(\alpha_{k_1}) \}) \cdot \mathbb{P}(\{ y_m = c_m : m \in \text{Child}(\alpha_{k_2}) \}),
\]

where \((\ast)\) follows from the independence between \(\alpha_{k_1}\) and \(\alpha_{k_2}\).

On the other hand, the above deduction also implies that if \(\{ y_j : j \in \text{Child}(\alpha_{k_1}) \}\) and \(\{ y_j : j \in \text{Child}(\alpha_{k_2}) \}\) are not independent, then

\[
\mathbb{P}(\{ y_j = c_j : j \in \text{Child}(\alpha_{k_1}) \}, \{ y_m = c_m : m \in \text{Child}(\alpha_{k_2}) \}) \\
- \mathbb{P}(\{ y_j = c_j : j \in \text{Child}(\alpha_{k_1}) \}) \cdot \mathbb{P}(\{ y_m = c_m : m \in \text{Child}(\alpha_{k_2}) \}) \\
= \sum_{(\alpha_{k_1}, \alpha_{k_2}) \in \{0,1\}^2} (\tilde{\nu}_{(\alpha_{k_1}, \alpha_{k_2})} - \tilde{\nu}_{\alpha_{k_1}} \tilde{\nu}_{\alpha_{k_2}}) \prod_{j \in \text{Child}(\alpha_{k_1})} \mathbb{P}(y_j \mid \alpha_{k_1}) \prod_{j \in \text{Child}(\alpha_{k_2})} \mathbb{P}(y_j \mid \alpha_{k_2}) \\
\neq 0
\]

for some \(\{ c_j : j \in \text{Child}(\alpha_{k_1}) \}\). This implies that there must exist some \((\alpha_{k_1}, \alpha_{k_2}) \in \{0,1\}^2\)
such that $\tilde{\nu}_{(\alpha_{k_1}, \alpha_{k_2})} - \tilde{\nu}_{\alpha_{k_1}} \tilde{\nu}_{\alpha_{k_2}} \neq 0$. This means $\alpha_{k_1} \not\perp \alpha_{k_2}$. The proof of the Proposition is complete. \hfill \Box

### S.2.5 Proof of Lemma 2

We use proof by contradiction. Suppose $\overline{\theta}_{c|0}^{(j)} = \theta_{c|1}^{(j)}$ for some $j$ and $c$. First consider $c < d$ then from the assumption (1) there is $\theta_{c|0}^{(j)} < \theta_{c|1}^{(j)}$. Then there are

$$
\sum_{\alpha : \alpha \geq g_j} \nu_{\alpha} \theta_{c|1}^{(j)} + \sum_{\alpha : \alpha \not\geq g_j} \nu_{\alpha} \theta_{c|0}^{(j)} \\
\bigg|^{(1)} < \theta_{c|1}^{(j)} = \overline{\theta}_{c|0}^{(j)} < \sum_{\alpha : \alpha \geq g_j} \overline{\nu}_{\alpha} \overline{\theta}_{c|1}^{(j)} + \sum_{\alpha : \alpha \not\geq g_j} \overline{\nu}_{\alpha} \overline{\theta}_{c|0}^{(j)}
$$

The above inequality contradicts the following fact implied by that $\overline{G}, \overline{\theta}, \overline{\nu}$ lead to the same distribution of the observed vector $y$.

$$
P(y_j = c \mid \overline{G}, \overline{\theta}, \overline{\nu}) = \sum_{\alpha \in \{0,1\}^K} \nu_{\alpha} \theta_{c|\alpha}^{(j)} = \sum_{\alpha \in \{0,1\}^K} \overline{\nu}_{\alpha} \overline{\theta}_{c|\alpha}^{(j)} = P(y_j = c \mid \overline{G}, \overline{\theta}, \overline{\nu}).
$$

This contradiction shows $\overline{\theta}_{c|0}^{(j)} \neq \theta_{c|0}^{(j)}$ must hold for any $1 \leq c < d - 1$. Similarly we can prove $\overline{\theta}_{d|0}^{(j)} \neq \theta_{d|0}^{(j)}$. By symmetry we also have $\overline{\theta}_{c|1}^{(j)} \neq \theta_{c|1}^{(j)}$ for all $j \in [p]$ and all $c \in [d]$. This proves Lemma 2. \hfill \Box

### S.2.6 Proof of Lemma 3

We next prove by contradiction. Assume there exists some $h \in [K]$ and a set $\mathcal{A} \subseteq [K] \setminus \{h\}$, such that

$$
\bigvee_{k \in \mathcal{A}} \tilde{g}_k \succeq \tilde{g}_h \tag{S.12}
$$

and also assume that there exists a set $\mathcal{B} \subseteq \{K+1, \ldots, J\}$ such that $\max_{m \in \mathcal{B}} g_{m,h} = 0$ and $\max_{m \in \mathcal{B}} g_{m,k} = 1$ for all $k \in \mathcal{A}$. 52
First, for each \( c \in \{1, \ldots, d - 1 \} \) define

\[
\Delta_{1^p, c} = \theta_{c, l}^{(h)} e_h + \sum_{k \in A} \theta_{c, k}^{(k)} e_k + \sum_{m=K+1}^p \theta_{c, m}^{(m)} e_m,
\]

(S.13)

\[
y_c^* = c \left( e_h + \sum_{k \in A} e_k + \sum_{m=K+1}^p e_m \right).
\]

(S.14)

Under the above definitions, we claim that the row vector of \( \bigotimes_{j \in [p]} \left( \Phi^{(j)} - \Delta_j \cdot 1_2^{K_j} \right) \) indexed by response pattern \( y_c^* \) is an all-zero vector. To see this, note that for any \( \alpha \in \{0, 1\}^K \), the corresponding element in the row denoted by \( t_{y^*_c, \alpha} \) contains a factor

\[
f_{\alpha} = \left( \bar{\theta}_{c, g}^{(h)} - \theta_{c, l}^{(h)} \right) \prod_{k \in A} \left( \bar{\theta}_{c, k}^{(k)} - \theta_{c, k}^{(k)} \right).
\]

This factor \( f_{\alpha} \) is potentially nonzero only if \( \theta_{c, g}^{(j)} \neq \theta_{c, l}^{(h)} \) and \( \bar{\theta}_{c, k}^{(k)} \neq \theta_{c, k}^{(k)} \) for all \( k \in A \) (equivalently, \( \bar{\theta}_{c, g}^{(k)} = \theta_{c, l}^{(k)} \) for all \( k \in A \)). However, this is impossible for any \( \alpha \) under the assumption (S.12) that \( \forall k \in A \; g_k \succeq g_h \). This is because for any \( \alpha \) such that \( \bar{\theta}_{c, g}^{(k)} = \theta_{c, l}^{(k)} \) for all \( k \in A \), there must be \( \alpha \succeq \forall k \in A \; g_k \), and our assumption (S.12) further gives \( \alpha \succeq g_h \), which means \( \bar{\theta}_{c, g}^{(k)} = \theta_{c, l}^{(k)} \). This proves \( f_{\alpha} = 0 \) must hold for all \( \alpha \in \{0, 1\}^K \). Since \( t_{y^*_c, \alpha} \) contains \( f_{\alpha} \) as a factor, there is \( t_{y^*_c, \alpha} = 0 \) for all \( \alpha \in \{0, 1\}^K \). Therefore \( \sum_{\alpha \in \{0, 1\}^K} t_{y^*_c, \alpha} \nu_{\alpha} = \sum_{\alpha \in \{0, 1\}^K} t_{y^*_c, \alpha} \nu_{\alpha} = 0 \). Now we focus on \( t_{y^*_c, \alpha} \). Note that \( G_{(K+1):2K} = I_K \). Due to the term \( \sum_{m=K+1}^p \theta_{c, m}^{(m)} e_m \) in the definition of \( \Delta \) in (S.13), we have \( t_{y^*_c, \alpha} \) is potentially nonzero only if \( \alpha = 1_K \). Therefore

\[
0 = \nu_{1_K} \left( \theta_{c, l}^{(h)} - \theta_{c, l}^{(h)} \right) \left( \theta_{c, l}^{(k)} - \theta_{c, l}^{(k)} \right) \prod_{m=K+1}^p \left( \theta_{c, m}^{(m)} - \theta_{c, m}^{(m)} \right).
\]

This gives \( \theta_{c, l}^{(h)} = \theta_{c, l}^{(h)} \).

Second, recall the set \( B \subseteq \{K + 1, \ldots, p\} \) defined earlier satisfies that \( \max_{m \in B} \; g_{m, h} = 0 \).
and \( \max_{m \in B} g_{m,k} = 1 \) for all \( k \in A \). For each \( c \in \{1, \ldots, d - 1\} \), now define

\[
\Delta_{1:p,c}^{**} = \bar{\theta}_{c|1}^{(h)} e_h + \sum_{k \in A} \bar{\theta}_{c|0}^{(k)} e_k + \sum_{m \in B} \theta_{c|0}^{(m)} e_m,
\]

\[(S.15)\]

\[
y_c^{**} = c \left( e_h + \sum_{k \in A} e_k + \sum_{m \in B} e_m \right).
\]

\[(S.16)\]

Under the above new definitions, we still claim that the row vector of \( \bigotimes_{j \in [p]} \left( \bar{\Phi}^{(j)} - \Delta_j \cdot 1_{2^K}^{\top} \right) \) indexed by response pattern \( y_c^{**} \) is an all-zero vector. The reasoning is similar to that in the previous paragraph after \((S.13)\), because that earlier argument only depends on the fact that \( \Delta_{1:p,c}^* \) contains the first two groups of terms \( \bar{\theta}_{c|1}^{(h)} e_h + \sum_{k \in A} \bar{\theta}_{c|0}^{(k)} e_k \), and \( \Delta_{1:p,c}^{**} \) also contains such two groups of terms. Therefore \( \sum_{\alpha \in \{0,1\}^K} t_{y_c^{**} \cdot \alpha} = \sum_{\alpha \in \{0,1\}^K} \bar{t}_{y_c^{**} \cdot \alpha} \bar{\theta}_c = 0 \).

Considering the \( \theta_c^{(h)} = \bar{\theta}_c^{(h)} \) obtained in the end of last paragraph, the element \( t_{y_c^{**} \cdot \alpha} \) would equal zero if \( \alpha_h = 1 \); this is because \( t_{y_c^{**} \cdot \alpha} \) contains a factor \( \bar{\theta}_c^{(h)} - \bar{\theta}_c^{(h)} \) which equals zero if \( \alpha_h = 1 \). This means the element \( t_{y_c^{**} \cdot \alpha} \) has the following property,

\[
t_{y_c^{**} \cdot \alpha} =
\begin{cases}
\left( \theta_c^{(h)} - \bar{\theta}_c^{(h)} \right) \prod_{k \in A} \left( \theta_c^{(k)} - \bar{\theta}_c^{(k)} \right) \prod_{m \in B} \left( \theta_c^{(m)} - \bar{\theta}_c^{(m)} \right), & \alpha_h = 0 \text{ and } \alpha \geq \bigvee_{m \in A \cup B} g_m; \\
0, & \text{otherwise}.
\end{cases}
\]

Now an important observation is that the following set \( \mathcal{M} \) of \( K \)-dimensional binary vectors is nonempty,

\[
\mathcal{M} := \left\{ \alpha \in \{0,1\}^K : \alpha_h = 0, \text{ and } \alpha \supseteq \bigvee_{m \in A \cup B} g_m \right\}.
\]

This is true because \( \max_{m \in B} g_{m,h} = 0 \) and \( \max_{m \in B} g_{m,k} = 1 \) for all \( k \in A \), and hence \( \alpha \supseteq \bigvee_{m \in A \cup B} g_m \) still allows for \( \alpha_h \) (that is, the \( h \)th element of \( \alpha \)) to be potentially zero. Now \( \sum_{\alpha \in \{0,1\}^K} t_{y_c^{**} \cdot \alpha} = 0 \) can be equivalently written as

\[
\left( \theta_{c|0}^{(h)} - \bar{\theta}_{c|1}^{(h)} \right) \prod_{k \in A} \left( \theta_{c|1}^{(k)} - \bar{\theta}_{c|0}^{(k)} \right) \prod_{m \in B} \left( \theta_{c|1}^{(m)} - \theta_{c|0}^{(m)} \right) \left( \sum_{\alpha \in \mathcal{M}} \nu_\alpha \right) = 0.
\]

\[(S.17)\]

Recall that \( \theta_{c|0}^{(j)} \neq \theta_{c|1}^{(j)} \) for all \( j \in [p] \) and \( \bar{\theta}_{c|0}^{(j)} \neq \bar{\theta}_{c|1}^{(j)} \) for all \( j \in [p] \), and also \( \sum_{\alpha \in \mathcal{M}} \nu_\alpha > 0 \). Therefore each factor of the left hand side of \((S.17)\) is nonzero, which gives a contradiction.

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This means the assumption (S.12) in the beginning of the proof is incorrect and the Lemma 3 is proved.

S.3 EM Algorithms for the BLESS Model

When G is known and fixed We first consider the scenario where the measurement graph \( G \) is known or already estimated, and describe the EM algorithm for the continuous parameters \( \theta \) and \( \nu \). Denote the subject-specific latent pattern indicators by \( z_{i,\alpha} = 1(\alpha_i = \alpha) \) and \( Z = (z_{i,\alpha}; \, i \in [N], \alpha \in \{0,1\}^K) \). An important observation is that the following equivalent formulation holds under the BLESS model,

\[
\theta_{c|\alpha}^{(j)} = \mathbb{1}(\alpha \succeq g_j)\theta_{c|1}^{(j)} + \left[1 - \mathbb{1}(\alpha \succeq g_j)\right]\theta_{c|0}^{(j)} = \left(\sum_{k=1}^{K} \alpha_k g_{j,k}\right)\theta_{c|1}^{(j)} + \left(1 - \sum_{k=1}^{K} \alpha_k g_{j,k}\right)\theta_{c|0}^{(j)}.
\]

Therefore, the complete data log-likelihood function under the BLESS model can be written as follows,

\[
\ell(\theta, \nu \mid Y, Z, G) = \sum_{i=1}^{N} \left\{ \sum_{\alpha \in \{0,1\}^K} z_{i,\alpha} \log(\nu_{\alpha}) + \sum_{j=1}^{d} \sum_{c=1}^{K} (y_{ijc} z_{i,\alpha}) \log(\theta_{c|\alpha}^{(j)}) \right\}
\]

\[
= \sum_{i=1}^{N} \sum_{\alpha \in \{0,1\}^K} \sum_{j=1}^{d} \sum_{c=1}^{K} y_{ijc} z_{i,\alpha} \left[ \sum_{k=1}^{K} \alpha_k g_{j,k} \log(\theta_{c|1}^{(j)}) + \left(1 - \sum_{k=1}^{K} \alpha_k g_{j,k}\right) \log(\theta_{c|0}^{(j)}) \right]
\]

\[
+ \sum_{\alpha \in \{0,1\}^K} \sum_{i=1}^{N} z_{i,\alpha} \log(\nu_{\alpha}).
\]

The above formulation allows for a convenient EM algorithm to compute the MLE, which iterates through E-step and a M-step towards convergence of the marginal log-likelihood. We present this EM algorithm in Algorithm 1.

When G is unknown We next describe a more general approximate EM algorithm that jointly estimate the \( G \) matrix and the continuous parameters. Introduce notation \( s = \)
Algorithm 1: EM algorithm for the BLESS Model when $G$ is Known

Data: Observed data array $Y = (y_{ijc})_{N \times p \times d} \in \{0, 1\}^{N \times p \times d}$ and number of latent variables $K$.

while not converged do
  // E Step
  Calculate the conditional expectation of each $z_{i, \alpha}$:
  $$
  E[z_{i, \alpha}] \leftarrow \frac{\nu_{\alpha} \prod_{j=1}^{p} \prod_{c=1}^{d} (\theta_{c|\alpha}^{(j)})^{y_{ijc}}}{\sum_{\alpha' \in \{0, 1\}^K} \nu_{\alpha'} \prod_{j=1}^{p} \prod_{c=1}^{d} (\theta_{c|\alpha'}^{(j)})^{y_{ijc}}}, \quad i \in [N], \quad \alpha \in \{0, 1\}^K.
  $$

  // M Step
  Update continuous parameters $\theta$ and $\nu$:
  $$
  \theta_{c|\alpha}^{(j)} \leftarrow \frac{\sum_{i=1}^{N} E[z_{i, \alpha}] \sum_{k=1}^{K} \alpha_k g_{j,k} y_{ijc}}{\sum_{i=1}^{N} E[z_{i, \alpha}] \sum_{k=1}^{K} \alpha_k g_{j,k}}, \quad j \in [p], \ c \in [d];
  $$
  $$
  \theta_{c|0}^{(j)} \leftarrow \frac{\sum_{i=1}^{N} E[z_{i, \alpha}] (1 - \sum_{k=1}^{K} \alpha_k g_{j,k}) y_{ijc}}{\sum_{i=1}^{N} E[z_{i, \alpha}] (1 - \sum_{k=1}^{K} \alpha_k g_{j,k})}, \quad j \in [p], \ c \in [d];
  $$
  $$
  \nu_{\alpha} \leftarrow \frac{\sum_{i=1}^{N} E[z_{i, \alpha}]}{\sum_{\alpha' \in \{0, 1\}^K} \sum_{i=1}^{N} E[z_{i, \alpha'}]}, \quad \alpha \in \{0, 1\}^K.
  $$

  Update $\theta_{c|\alpha}^{(j)} = \mathbb{1}(\alpha \geq g_j) \theta_{c|\alpha}^{(j)} + (1 - \mathbb{1}(\alpha \geq g_j)) \theta_{c|0}^{(j)}$ after completing the M Step.

Output: Parameters $\theta$, $\nu$.

$(s_1, \ldots, s_p)$ with each $s_j \in [K]$, where $s_j = k$ if $g_{j,k} = 1$. Then there is a one-to-one correspondence between the vector $s$ and matrix $G$. We can just augment the EM algorithm described above by adding the following step of drawing samples of $\{g_{j,k}\}$ in the E step. The conditional distribution of each $s_j$ is the Categorical distribution with parameters as follows,

$$
\gamma_{j,k} = P(s_j = k \mid -) = \frac{\prod_{\alpha} \prod_{i=1}^{N} \prod_{c=1}^{d} [(\theta_{c|1}^{(j)})^{\alpha_k} (\theta_{c|0}^{(j)})^{1-\alpha_k}]^{y_{ijc}z_{i,\alpha}}}{\sum_{k'=1}^{K} \prod_{\alpha} \prod_{i=1}^{N} \prod_{c=1}^{d} [(\theta_{c|1}^{(j)})^{\alpha_{k'}} (\theta_{c|0}^{(j)})^{1-\alpha_{k'}}]^{y_{ijc}z_{i,\alpha}}} = \frac{\prod_{c=1}^{d} [(\theta_{c|1}^{(j)})^{\alpha_k} (\theta_{c|0}^{(j)})^{1-\alpha_k}]^{\sum_{i=1}^{N} y_{ijc}z_{i,\alpha}}}{\sum_{k'=1}^{K} \prod_{c=1}^{d} [(\theta_{c|1}^{(j)})^{\alpha_{k'}} (\theta_{c|0}^{(j)})^{1-\alpha_{k'}}]^{\sum_{i=1}^{N} y_{ijc}z_{i,\alpha}}}.
$$

Since the entries of the $G$ are needed in the E step of the algorithm, after obtaining the $\gamma_{j,k}$, we let $s_j = k$ if the current posterior probability $P(s_j = k \mid -)$ is the largest among all the $K$ posterior probabilities. Such a procedure has a similar spirit to a classification EM algorithm (Celeux and Govaert, 1992), but the difference is that we use this procedure to...
update the graphical structure (the entries of the measurement graph), instead of updating the subject-specific latent variables as in classification EM. We present this general EM algorithm dealing with unknown $\mathbf{G}$ in Algorithm 2.

**Algorithm 2:** Approximate EM algorithm for the BLESS Model when $\mathbf{G}$ is Unknown

**Data:** Observed data array $\mathbf{Y} = (y_{ijc})_{N \times p \times d} \in \{0, 1\}^{N \times p \times d}$ and number of latent variables $K$.

**while not converged do**

```
// E Step
Calculate the conditional expectation of each $z_{i,\alpha}$:

$$
\mathbb{E}[z_{i,\alpha}] = \mathbb{P}(a_i = \alpha | -) \left\langle \frac{\nu_{\alpha} \prod_{j=1}^{p} \prod_{c=1}^{d} (\theta_{j,c|\alpha}^{(j)})^{y_{jic}}}{\sum_{\alpha' \in \{0,1\}^K} \nu_{\alpha'} \prod_{j=1}^{p} \prod_{c=1}^{d} (\theta_{j,c|\alpha'}^{(j)})^{y_{jic}}} \right\rangle, \ i \in [N], \ \alpha \in \{0, 1\}^K.
$$

Draw each $a_i$ from the above Categorical distribution with $2^K$ components. For each $j \in [p]$ and $k \in [K]$, let

$$
\gamma_{j,k} \left\langle \frac{\prod_{\alpha} \prod_{c=1}^{d} \left( (\theta_{j,c|1}^{(j)})^{\alpha_k} (\theta_{j,c|0}^{(j)})^{1-\alpha_k} \right)^{\sum_{i=1}^{N} y_{jic} z_{i,\alpha}}} {\sum_{K'}^{K} \prod_{\alpha} \prod_{c=1}^{d} \left( (\theta_{j,c|1}^{(j)})^{\alpha_{K'}'} (\theta_{j,c|0}^{(j)})^{1-\alpha_{K'}'} \right)^{\sum_{i=1}^{N} y_{jic} z_{i,\alpha}}}, \ g_{j,k} \left\langle 1 \right. \text{ if } \gamma_{j,k} = \max\{\gamma_{j,1}, \ldots, \gamma_{j,K}\}; \ g_{j,k} \left\langle 0 \right. \text{ otherwise.}
$$

// M Step
Update continuous parameters $\theta$ and $\nu$:

$$
\theta_{j,c|1}^{(j)} \left\langle \frac{\sum_{\alpha} \sum_{i=1}^{N} \mathbb{E}[z_{i,\alpha}] \sum_{k=1}^{K} \alpha_k g_{j,k} y_{jic}} {\sum_{\alpha} \sum_{i=1}^{N} \mathbb{E}[z_{i,\alpha}] \sum_{k=1}^{K} \alpha_k g_{j,k}}, \ j \in [p], \ c \in [d];
$$

$$
\theta_{j,c|0}^{(j)} \left\langle \frac{\sum_{\alpha} \sum_{i=1}^{N} \mathbb{E}[z_{i,\alpha}] (1 - \sum_{k=1}^{K} \alpha_k g_{j,k}) y_{jic}} {\sum_{\alpha} \sum_{i=1}^{N} \mathbb{E}[z_{i,\alpha}] (1 - \sum_{k=1}^{K} \alpha_k g_{j,k})}, \ j \in [p], \ c \in [d];
$$

$$
\nu_{\alpha} \left\langle \frac{\sum_{i=1}^{N} \mathbb{E}[z_{i,\alpha}]} {\sum_{\alpha' \in \{0,1\}^K} \sum_{i=1}^{N} \mathbb{E}[z_{i,\alpha'}]}, \ \alpha \in \{0, 1\}^K.
$$

Update $\theta_{j,c|\alpha}^{(j)} = 1(\alpha \geq g_{j})\theta_{j,c|1}^{(j)} + (1 - 1(\alpha \geq g_{j}))\theta_{j,c|0}^{(j)}$ after completing the M Step.

**Output:** Measurement graph $\mathbf{G}$ and parameters $\theta$, $\nu$.\"