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OPTION BOUNDS

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Abstract

In this paper, we obtain sharp estimates for the expected payoffs and prices of European call options on an asset with an absolutely continuous price in terms of the price density characteristics. These techniques and results complement other approaches to the derivative pricing problem. Exact analytical solutions to option-pricing problems and to Monte-Carlo techniques make strong assumptions on the underlying asset's distribution. In contrast, our results are semi-parametric. This allows the derivation of results without knowing the entire distribution of the underlying asset's returns. Our results can be used to test different modelling assumptions. Finally, we derive bounds in the multiperiod binomial option-pricing model with time-varying moments. Our bounds reduce the multiperiod setup to a two-period setting, which is advantageous from a computational perspective.

Keywords: Option pricing; semi-parametric bounds

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1. Introduction

Methodologies for asset pricing and contingent claim pricing belong to one of three main ideologies: simulations, closed analytical solutions and bounds. Monte-Carlo techniques are the most widely used simulation methodology. To use Monte-Carlo simulations, it is necessary to make very strong assumptions concerning the underlying distribution. As the distribution family and its parameters are assumed known, this technique falls under the heading of parametric estimation. Another heading, semi-parametric estimation, only assumes that a specific number of characteristics, such as moments, are known. This second approach helps to reduce the modelling error inherent in the parametric approach. Which estimation technique will be more exact depends on how accurate the parametric assumptions are. In any case, the results of a semi-parametric estimate should be more robust.

Due to the complicated structure of the asset-pricing problem, when deriving exact analytical solutions, it is usually necessary to make restrictive assumptions or to use simplifying approximations. Both of these approaches, which are used to make the problem tractable analytically, introduce modelling error into the closed-form solution. Unfortunately, the properties of these introduced modelling errors are not usually easy to study and analyse. Most importantly, neither do they lend themselves to the study of the error propagation. Without such error bounds it is difficult to ascertain the validity of a model and its assumptions.

The bounds approach gives upper and lower bounds to such errors and thus provides an indirect test of model misspecification for the exact methodologies. In fact, Grundy (1991) noted that the problem can be inverted and estimates for bounds on the parameters of the assumed distribution can be inferred from the bounds' formulae using observed prices. Thus it is important to have bounds on the true price in a distribution-free setting, so that we can have a true range of possible values. We derive such bounds. For vanilla options, our bounds are improvements on those of Lo (1987) and Grundy (1991) whose bounds are sharp for a two-point process.

We develop bounds that are sharp for a continuous uniform process. We then extend the results to include higher moments. With the standard risk-averse agent under expected utility theory, third moments naturally have important implications. If utility for the second moment is negative, as implied by risk aversion, then marginal utility must go to zero as price grows otherwise the utility curve will eventually turn down. This implies positive skewness in the asset-return distribution, otherwise agents will dislike large amounts of money, which is counterintuitive. Empirically there is a large body of literature supporting the importance of skewness for the agent's decision process. Sun and Yan (2003) studied the effects of skewness in the portfolio decision problem and find that skewness is traded off for mean return and extra variance will be taken if the distribution has positive skewness. Their paper contains an extensive reference to this literature.

As option-like assets exist in almost any investment portfolio, whether owned by an individual or a corporation, there is wide interest and a large financial literature on option pricing and option bounds. Perrakis and Ryan (1984) and Perrakis (1986) developed option bounds in discrete time. Although the results in these papers are quite general, they assume that the whole distribution of returns is known. Lo (1987) and Grundy (1991) extended the option-bound results to semi-parametric formulae and thus considerably weakened the necessary assumptions to apply their bounds. Grundy applied these results to obtain lower bounds on the noncentral moments of the underlying asset's return distribution when option prices are observed. Boyle and Lin (1997) extended Lo's results to contingent claims based on multiple assets. Constantinides and Zariphopoulou (2001) studied intertemporal bounds under transaction costs. Frey and Sin (1999) studied bounds under a stochastic volatility model. Bertsimas and Popescu (2002) derived bounds that include information that is included in related assets and they extended their results for the effects of transaction costs. The problem of determining sharp option bounds is closely related to the problem of evaluating the extrema of a firm's expected profit in inventory theory (see Scarf (1958), (2002)).

The results in this paper complement this option-bounds literature. We obtain sharp bounds for a European call option's expected payoff and current price. The underlying asset has an absolutely continuous price which can be expressed in terms of the price density characteristics, such as its L_∞ and L_p norms.

The binomial option model is currently the most widely used model in Wall Street. Cox *et al.* (1979) introduced the binomial option-pricing methodology and demonstrated its convergence properties. Stapleton and Subrahmanyam (1984) showed that, under certain preference assumptions, the binomial methodology is valid even if opportunities to hedge do not exist. Boyle and Vorst (1992) derived self-financing strategies that perfectly replicate the final payoffs to long and short positions in calls and puts assuming proportional transaction costs on trades in the stock and no transaction costs in the bond. Palmer (2001) extended Boyle and Vorst's results to general transaction costs by relaxing their assumptions.

In addition to the semi-parametric bounds on option prices, we also obtain sharp bounds on expected payoffs in the multiperiod binomial model. (Britten-Jones and Neuberger (2000) derived a volatility forecast in continuous time that is model-free. Their results also give a range of prices for an option.) This bound reduces the problem of approximating the expected payoff in the multiperiod setup to the two-period case. There has been little work on time-varying moments within the framework of the binomial model. (Dumas *et al.* (1998) showed that deterministic volatility models perform poorly out-of-sample; Buraschi and Jackwerth (2001) presented evidence that volatility is stochastic. Ball and Roma (1994) and Frey (1997) surveyed stochastic volatility in continuous time.) Our model extends the functionality of such an approach. Our bounds on the multiperiod binomial model allow the probability of an up move to change with time; thus one time-varying moment, such as stochastic volatility or time-varying risk premiums, can be accommodated. Our binomial option bounds complement and can be used to test stochastic volatility option models for misspecification error. (Given our extension of the standard binomial model only one moment can be matched. The structure of the binomial model links the first two moments, e.g. if stochastic volatility is matched, then the return is set as well.)

Finally, bounds for options are important in that there is a result that shows that, for derivatives with payoffs bounded below, the minimum initial value of a self-financing strategy that superreplicates the payoff of a derivative equals the supremum of the expected value of the terminal payoff under all equivalent martingale measures (see Delbaen (1992), El Karoui and Quenez (1995) and Kramkov (1996)). It is well known that stochastic volatility models are incomplete. Thus, the model admits many equivalent martingale measures. Combined with bounds over all risk-neutral distributions, this result allows pricing of the contingent claim in the presence of stochastic volatility.

The paper is organized as follows. In Section 2, we derive sharp bounds in terms of the norms and tail probabilities of the density of the underlying asset's price. In Section 3, we derive our results for the binomial model. In Section 4, we make some concluding remarks.

2. Sharp bounds on the expected payoffs and prices of European call options

Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_t$ be a sequence of σ-algebras on a probability space $(\Omega, \mathcal{F}_t, \mathrm{P}_t)$. Throughout the paper, we deal with a complete and arbitrage-free securities market consisting of two assets. One asset is the risky asset with price $S_t \geq 0$ for $t \geq 0$. The sequence (S_t) is adapted to the sequence of information sets $(\mathcal{F}_t)$ and the nonnegativity constraint reflects the limited-liability condition inherent in a contingent claim. The second asset is a money-market account with a risk-free rate of return r. In what follows, $\mathrm{E}_t[\cdot] = \mathrm{E}_t[\cdot \mid \mathcal{F}_t]$ denotes the day-t conditional expectation and $\mathrm{P}_t = \mathrm{P}_t(\cdot \mid \mathcal{F}_t)$ denotes the day-t conditional probability.

Let us begin by recalling the results on an option's expected payoff and current price obtained by Lo (1987) and Grundy (1991). Consider a European call option on the risky asset with strike price K and expiring at time T. The day-t expected payoff of the option is $\mathrm{E}_t \max(S_T - K, 0)$ and its day-t risk-neutral price is $\mathrm{e}^{-r(T-t)}\,\mathrm{E}_t^* \max(S_T - K, 0)$, where E_t^* denotes the expectation with respect to the unique equivalent probability measure. Lo (1987) showed that the day-t expectation $\mathrm{E}_t \max(S_T - K, 0)$ satisfies the following sharp inequalities:

$$\mathrm{E}_t \max(S_T - K, 0) \leq \begin{cases} \mu_t - K\dfrac{\mu_t^2}{\sigma_t^2 + \mu_t^2} & \text{if } K \leq \dfrac{\sigma_t^2 + \mu_t^2}{2\mu_t}, \\ \tfrac{1}{2}[\mu_t - K + \sqrt{(K - \mu_t)^2 + \sigma_t^2}] & \text{otherwise,} \end{cases} \tag{1}$$

where

$$\mu_t = \mathrm{E}_t\, S_T \quad \text{and} \quad \sigma_t^2 = \mathrm{E}_t\, S_T^2 - \mu_t^2.$$

Grundy (1991) proved that the following sharp estimates hold for $p \geq 1$:

$$\mathrm{E}_t \max(S_T - K, 0) \leq \begin{cases} (\mathrm{E}_t\, S_T^p)^{1/p} - K & \text{if } K \leq \dfrac{p-1}{p}(\mathrm{E}_t\, S_T^p)^{1/p}, \\ (\mathrm{E}_t\, S_T^p)\dfrac{1}{p}\left(\dfrac{p-1}{pK}\right)^{p-1} & \text{otherwise.} \end{cases} \tag{2}$$

Suppose that, in addition to the knowledge of the mean $\mu_t = \mathrm{E}_t\, S_T$ and the variance $\sigma_t^2 = \mathrm{E}_t\, S_T^2 - \mu_t^2$, as in Lo's (1987) setup, we also know the day-t tail probability $\pi_t = 1 - F_t(K) = \mathrm{P}_t(S_T > K)$ of the terminal asset price at only one point (or, equivalently, the day-t probability that the call option will finish in the money). When a binomial model is appropriate, π_t can be estimated using Theorem 5 below. The following theorem gives easily computable sharp bounds for the expected payoff of the European call option in terms of the characteristics μ_t, σ_t^2 and π_t of the underlying asset-price distribution.

Theorem 1. *The following sharp inequalities hold:*

$$(\mu_t - K)\pi_t \leq \mathrm{E}_t \max(S_T - K, 0) \leq (\mu_t - K)\pi_t + \sigma_t\sqrt{\pi_t - \pi_t^2}. \tag{3}$$

Remark 1. The bounds (3) are, in fact, inequalities between the expected payoffs of the European call option and the cash-or-nothing option on the same asset, where the cash-or-nothing option pays $\mu_t - K$ if the price of the asset is greater than the strike price K of the call option at time T and pays 0 otherwise. The left-hand side of (3) means that the fair price of the cash-or-nothing option is no greater than that of the call option. The right-hand side of (3) adjusts the payoff of the cash-or-nothing option by a risk premium that reflects the variance of the payoff to the option, $\sigma^2 = \mathrm{var}(S_T - K)$, and the variance of the probability of actually receiving the payoff, $\sqrt{\pi_t - \pi_t^2} = \mathrm{var}[\mathbf{1}(S_T > K)]$, where $\mathbf{1}(\cdot)$ denotes the indicator function.

Proof of Theorem 1. For all random variables X and Y,

$$\mathrm{E}_t\, XY = \mathrm{E}_t\, X\, \mathrm{E}_t\, Y + \tfrac{1}{2}\, \mathrm{E}_t(X - X')(Y - Y'), \tag{4}$$

where the random vector (X', Y') is an independent copy of (X, Y) (and so has the same distribution), provided the expectations exist. (A functional analogue of (4) was used by Barnett *et al.* (2001) to obtain bounds for the variance of a random variable with a finite support.) Let S_T' be an independent copy of S_T. Taking $X = S_T - K$, $Y = \mathbf{1}(S_T > K)$, $X' = S_T' - K$ and $Y' = \mathbf{1}(S_T' > K)$ in (4), we get

$$\begin{aligned} &\mathrm{E}_t \max(S_T - K, 0) \\ &\quad = \mathrm{E}_t(S_T - K)\,\mathbf{1}(S_t > K) \\ &\quad = (\mathrm{E}_t\, S_T - K)\,\mathrm{E}_t\,\mathbf{1}(S_T > K) + \tfrac{1}{2}\,\mathrm{E}_t[\{S_T - S_T'\}\{\mathbf{1}(S_T > K) - \mathbf{1}(S_T' > K)\}] \\ &\quad = (\mu_t - K)\pi_t + \tfrac{1}{2}\,\mathrm{E}_t[(S_T - S_T')(\mathbf{1}(S_T > K) - \mathbf{1}(S_T' > K))] \quad \text{a.s.} \end{aligned} \tag{5}$$

Since, as it is easy to see, $\mathrm{E}_t[\{S_T - S_T'\}\{\mathbf{1}(S_T > K) - \mathbf{1}(S_T' > K)\}] \geq 0$, we get the left-hand inequality of (3). By Hölder's inequality,

$$\begin{aligned} &\mathrm{E}_t[\{S_T - S_T'\}\{\mathbf{1}(S_T > K) - \mathbf{1}(S_T' > K)\}] \\ &\quad \leq [\mathrm{E}_t(S_T - S_T')^2]^{1/2}[\mathrm{E}(\mathbf{1}(S_T > K) - \mathbf{1}(S_T' > K))^2]^{1/2} \\ &\quad = 2\sigma_t\sqrt{\pi_t - \pi_t^2}. \end{aligned}$$

This and (5) imply the upper bound in (3).

To show that the upper bound in (3) is sharp, take $K = \sqrt{3}/3$ and the random variable S_T uniformly distributed on $[0, 2\sqrt{3}/3]$. Then $\mu = \mathrm{E}\, S_T = \sqrt{3}/3$, $\pi = \mathrm{P}(S_T > K) = \frac{1}{2}$, $\sigma^2 = \mathrm{E}\, S_T^2 - \mu^2 = \frac{1}{9}$ and

$$\mathrm{E} \max(S_T - K, 0) = \tfrac{1}{6} = (\mu - K)\pi + \sigma\sqrt{\pi - \pi^2}.$$

Sharpness of the lower bound in (3) follows, for example, from the choice $S_T = 1$ almost surely and $K = \frac{1}{2}$. The proof is complete.

Let now $0 \le a_t < b_t$, $T \ge 1$ and suppose that the asset price S_t takes values in the interval $[a, b]$. Suppose further that the price S_T is absolutely continuous. Denote by f_t the price's conditional density and by $F_t(x) = \int_{-\infty}^{x} f_t(s)\, \mathrm{d}s$ its conditional distribution function with respect to $\mathcal{F}_t$, the time-t information set.

The following theorem gives bounds on the day-t expected payoff of the European call option with strike price K on the asset expiring at time T.

Theorem 2. *The following sharp inequalities hold:*

1. *provided that $f_t \in L_\infty[a, b]$ (conditionally on $\mathcal{F}_t$),*

$$\mathrm{E}_t \max(S_T - K, 0) \le (\mu_t - K)\pi_t + \frac{b-a}{2}(b - K)(K - a)\|f_t\|_\infty^2\, \mathbf{1}(a < K < b); \tag{6}$$

2. *provided that $f_t \in L_p[a, b]$ (conditionally on $\mathcal{F}_t$), $p > 1$, $1/p + 1/q = 1$,*

$$\begin{aligned}\mathrm{E}_t &\max(S_T - K, 0)\\ &\le (\mu_t - K)\pi_t\\ &\quad + \|f_t\|_p^2 \left(\frac{(b-a)^{q+2} - (b-K)^{q+2} - (K-a)^{q+2}}{(q+1)(q+2)}\right)^{1/q} \mathbf{1}(a < K < b);\end{aligned} \tag{7}$$

3. *and*

$$\mathrm{E}_t \max(S_T - K, 0) \le (\mu_t - K)\pi_t + (b - a)\, \mathbf{1}(a < K < b). \tag{8}$$

Remark 2. The upper bound in (3) and the estimates (6)–(8) can be better than those obtained by Lo (1987) and Grundy (1991). For example, if S_T is uniformly distributed on $[0, b]$ and $K = b/3$, then the upper bound in (3) gives the estimate $\mathrm{E}_t \max(S_T - K, 0) \le 0.247b$, while it follows from (1) and (2) that $\mathrm{E}_t \max(S_T - K, 0) \le 0.250b$ and the true value of $\mathrm{E}_t \max(S_T - K, 0)$ is $0.222b$. As will be noted in the proof of Theorem 2, in the case of a uniformly distributed random variable S_T, the estimate (6) holds as equality and the bound (7) holds as equality in the limit. For example, for S_T uniformly distributed on $[0, b]$ and $K = 0.75b$, we get $\mathrm{E}_t \max(S_T - K, 0) = 0.031b = (\mathrm{E}_t\, S_T - K)(1 - F_t(K)) + (b/2)(b - K)(K - a)\|f_t\|_\infty^2$. In this case, the estimate (7) with $p = 11$ gives the estimate $\mathrm{E}_t \max(S_T - K, 0) \le 0.048b$, while it follows from (1) that $\mathrm{E}_t \max(S_T - K, 0) \le 0.066b$ and (2) gives the estimate $\mathrm{E}_t \max(S_T - K, 0) \le 0.111b$.

Remark 3. The structure of the bounds given by Theorems 1 and 2 is simpler than that of the bounds in Lo (1987) and Grundy (1991) in the sense that the form of the estimates in the theorems is the same for all values of K, the option strike price, while the forms of the bounds obtained by the above authors are different from each other in the cases of small and large K.

Proof of Theorem 2. It is evident that the inequalities (6)–(8) hold for $K \le a$ and $K \ge b$. Let $a < K < b$. Using (4) with $X = S_T - K$, $Y = \mathbf{1}(S_T > K)$, $X' = S'_T - K$ and $Y' = \mathbf{1}(S'_T > K)$, we get

$$\begin{aligned}
&\mathrm{E}_t \max(S_T - K, 0) - \mathrm{E}_t(S_T - K)\,\mathrm{P}(S_T > K)\\
&\quad = \mathrm{E}_t \max(S_T - K, 0) - (\mu_t - K)\pi_t\\
&\quad = \int_a^b f_t(s)\,\mathrm{d}s \int_a^b (s-K)\,\mathbf{1}(t > K) f_t(s)\,\mathrm{d}s\\
&\qquad - \int_a^b (s-K) f_t(s)\,\mathrm{d}s \int_a^b \mathbf{1}(s > K) f_t(s)\,\mathrm{d}s\\
&\quad = \frac{1}{2}\int_a^b\int_a^b (s-u)[\mathbf{1}(s > K) - \mathbf{1}(u > K)] f_t(s) f_t(u)\,\mathrm{d}s\,\mathrm{d}u. \qquad (9)
\end{aligned}$$

Suppose that $f_t \in L_\infty[a, b]$ (conditionally on $\mathcal{F}_t$). Then it follows from (9) that

$$\begin{aligned}
&\mathrm{E}_t \max(S_T - K, 0) - (\mu_t - K)\pi_t\\
&\quad \le \frac{1}{2}\Big(\sup_{(s,u)\in[a,b]^2} f_t(s) f_t(u)\Big)\int_a^b\int_a^b (s-u)[\mathbf{1}(s > K) - \mathbf{1}(u > K)]\,\mathrm{d}s\,\mathrm{d}u\\
&\quad = \frac{1}{2}\| f_t \|_\infty^2 \int_a^b\int_a^b (s-u)(\mathbf{1}(s > K) - \mathbf{1}(u > K))\,\mathrm{d}s\,\mathrm{d}u. \qquad (10)
\end{aligned}$$

Now, since

$$\begin{aligned}
\int_a^b\int_a^b (s-u)[\mathbf{1}(s > K) - \mathbf{1}(u > K)]\,\mathrm{d}s\,\mathrm{d}u &= 2\int_K^b\int_a^K (s-u)\,\mathrm{d}s\,\mathrm{d}u\\
&= (b-a)(b-K)(K-a),
\end{aligned}$$

from (10) we obtain that (6) holds for $a < K < b$. Let now $f_t \in L_p[a, b]$ (conditionally on $\mathcal{F}_t$), $p > 1$, $1/p + 1/q = 1$. By Hölder's inequality, we have

$$\begin{aligned}
&\int_a^b\int_a^b f_t(s) f_t(u)(s-u)(\mathbf{1}(s > K) - \mathbf{1}(u > K))\,\mathrm{d}s\,\mathrm{d}u\\
&\quad = 2\int_K^b\int_a^K f_t(s) f_t(u)(s-u)\,\mathrm{d}s\,\mathrm{d}u\\
&\quad \le 2\Big(\int_a^b\int_a^b f_t^p(s) f_t^p(u)\,\mathrm{d}s\,\mathrm{d}u\Big)^{1/p}\Big(\int_K^b\int_a^K (s-u)^q\,\mathrm{d}s\,\mathrm{d}u\Big)^{1/q}\\
&\quad = 2\| f_t \|_p^2\Big(\frac{(b-a)^{q+2} - (b-K)^{q+2} - (K-a)^{q+2}}{(q+1)(q+2)}\Big)^{1/q}. \qquad (11)
\end{aligned}$$

The relations (9) and (11) imply that (7) holds for $a < K < b$. Since

$$\begin{aligned}
&\int_a^b\int_a^b f_t(s) f_t(u)(s-u)(\mathbf{1}(s > K) - \mathbf{1}(u > K))\,\mathrm{d}s\,\mathrm{d}u\\
&\qquad \le \sup_{(s,u)\in[a,b]^2} (s-u)(\mathbf{1}(s > K) - \mathbf{1}(u > K))\\
&\qquad = b - a,
\end{aligned}$$

from (9) we get the inequality (8) for $a < K < b$. It is easy to see that the bound (6) holds as equality for the random variable S_T uniformly distributed on $[a, b]$ and $K \in (a, b)$:

$$\mathrm{E}_t \max(S_T - K, 0) = \frac{(b-K)^2}{2(b-a)} = (\mathrm{E}_t\, S_T - K)(1 - F_t(K)) + \left(\frac{b-a}{2}\right)(b-K)(K-a)\|f_t\|_\infty^2.$$

Furthermore, (7) gives in the above case the estimate

$$\frac{(b-K)^2}{2(b-a)} \leq \left(\frac{a+b}{2} - K\right)\frac{b-K}{b-a} + (b-a)^{2/p-2}\left[\frac{(b-a)^{q+2} - (b-K)^{q+2} - (K-a)^{q+2}}{(q+1)(q+2)}\right]^{1/q}$$

for all $p > 1$, $1/p + 1/q = 1$. Letting $p \to \infty$, we get the equality, which proves sharpness of (7). The proof is complete.

The following theorem gives bounds on the expected payoff of a European call option in terms of the first three moments of the underlying asset's distribution. Let us introduce additional notation: $\mu_{1t} = \mathrm{E}_t\, S_T$, $\mu_{2t} = \mathrm{E}_t\, S_T^2$, $\mu_{3t} = \mathrm{E}_t\, S_T^3$, $\mu'_{2t} = \sigma_t^2 = \mu_{2t} - \mu_{1t}^2$, $\mu'_{3t} = \mu_{3t} - 3\mu_{1t}\mu_{2t} + 2\mu_{1t}^3$. Let

$$c_t = \mu_{1t} + \frac{\mu_{3t} - \sqrt{\mu_{3t}'^2 + 4\mu_{2t}'^3}}{2\mu'_{2t}},$$

$$c'_t = \mu_{1t} + \frac{\mu_{3t} + \sqrt{\mu_{3t}'^2 + 4\mu_{2t}'^3}}{2\mu'_{2t}},$$

$$r = K - \sqrt{(\mu_{1t} - K)^2 + \mu_{2t} - \mu_{1t}^2}.$$

Denote by s the largest (real) root in the interval $[2c_t'^2/(3c'_t - c_t), \infty)$ of the cubic equation

$$-2\mu_{1t}s^3 + (2\mu_{2t} + 3\mu_{1t}K)s^2 - 2\mu_{2t}Ks + K\mu_{3t} = 0.$$

Set $s' = (\mu_{3t} - s\mu_{2t})/(\mu_{2t} - s\mu_{1t})$.

Theorem 3. *The following sharp inequalities hold:*

$$\mathrm{E}_t \max(S_T - K, 0) \leq \begin{cases} \mu_{1t} - K\dfrac{\mu_{1t}^2}{\mu_{2t}} & \text{if } K \leq \dfrac{\mu_{2t}}{2\mu_{1t}}, \\ \dfrac{(r-\mu_{1t})^2}{r^2 - 2r\mu_{1t} + \mu_{2t}}\left(\dfrac{r\mu_{1t} - \mu_{2t}}{r - \mu_{1t}} - K\right) & \text{if } \dfrac{\mu_{2t}}{2\mu_{1t}} \leq K \leq \mu_{1t} + \dfrac{\mu'_{3t}}{2\mu'_{2t}}, \\ \dfrac{\mu_{1t} - c_t}{c'_t - c_t}(c'_t - K) & \text{if } \mu_{1t} + \dfrac{\mu'_{3t}}{2\mu'_{2t}} \leq K \leq \dfrac{2c_t'^2}{3c'_t - c_t}, \\ \dfrac{\mu_{2t} - s'\mu_{1t}}{s(s-s')}(s - K) & \text{if } K \geq \dfrac{2c_t'^2}{3c'_t - c_t}. \end{cases}$$

Proof. Denote by $\phi(\mu_1, \mu_2, \mu_3)$ the right-hand side of the inequality in the theorem. For any random variable $X \geq 0$ with $\mathrm{E}\, X = \mu_1$, $\mathrm{E}\, X^2 = \mu_2$ and $\mathrm{E}\, X^3 = \mu_3$ there exists a sequence of random variables X_n with bounded support $0 \le X_n \le b_n$, $b_n \to \infty$, such that $\mathrm{E}\, X_n = \mu_1$, $\mathrm{E}\, X_n^2 = \mu_2$ and $\mathrm{E}\, X_n^3 = \mu_3$. According to Jansen *et al.* (1986),

$$\mathrm{E}\max(X_n - K, 0) \le \psi(\mu_1, \mu_2, \mu_3, b_n), \tag{12}$$

where

$$\lim_{b\to\infty} \psi(\mu_1, \mu_2, \mu_3, b) = \phi(\mu_1, \mu_2, \mu_3). \tag{13}$$

Since the sequence $\{\max(X_n - K, 0)\}_{n=1}^{\infty}$ is obviously uniformly integrable, taking limits in (12) implies that

$$\mathrm{E}\max(X - K, 0) \le \phi(\mu_1, \mu_2, \mu_3). \tag{14}$$

Furthermore, by Jansen *et al.* (1986), there exists a sequence of random variables $\{X_n\}_{n=1}^{\infty}$, $0 \le X_n \le b_n$, $b_n \to \infty$, such that $\mathrm{E}\, X_n = \mu_1$, $\mathrm{E}\, X_n^2 = \mu_2$, $\mathrm{E}\, X_n^3 = \mu_3$ and $\mathrm{E}\max(X_n - K, 0) = \phi(\mu_1, \mu_2, \mu_3, b_n)$. This and (13) prove the sharpness of the inequality (14). This completes the proof.

Applying the fact that the call option price c equals its expected payoff at maturity when discounted at the expected return on the option over its lifetime (see e.g. Grundy (1991)), it follows that

$$c = \frac{\text{expected payoff}}{1 + \text{expected return}},$$

where the expectations are with respect to the true probability measure. By applying lower bounds on the discount rate, we can use an observed option price and Theorems 1–3 (for expectations under the true probability measure) to immediately obtain lower bounds on the underlying asset's distributional characteristics. Perrakis and Ryan (1984) and Grundy (1991) derived several conditions under which the mean return on the asset gives a lower bound on the expected return on the option over its lifetime. (The sufficient conditions are given in Proposition 6 of Grundy (1991). If interest rates are nonstochastic and the underlying asset's price dynamics are given by the diffusion equation $\partial S_t = \alpha(t) S_t \,\mathrm{d}t + \sigma(S_t, t) S_t \,\mathrm{d}Z_t$ with $\alpha(\cdot) \ge r(t)$ for all t and $\partial c/\partial t \le 0$, then the expected return on a call over its life is always at least μ_1.) For example, under those conditions, the following results (which immediately follow from Theorems 1 and 2 and are similar to those in Grundy (1991)) can be used to derive information about the parameters of the underlying asset's true distribution. If the expected return on a call option with strike price K is at least as large as the expected return on the underlying asset and the option is trading at price c, then all feasible triples of the underlying asset price's mean μ, variance σ and probability of the option finishing at the money p calculated under the true probability distribution satisfy

$$c \le \frac{(\mu_t - K)\pi + \sigma\sqrt{\pi - \pi^2}}{\mu_t}.$$

Under the same conditions, if the asset's price takes values in the interval $[a, b]$ and is absolutely continuous, then the values μ, π and the norms of the price density f under the true probability measure satisfy

$$c \le \frac{(\mathrm{E}_t\, S_t - K)\pi + ((b-a)/2)(b-K)(K-a)\|f_t\|_\infty^2\, \mathbf{1}(a < K < b)}{\mu_t}$$

provided that $f_t \in L_\infty[a, b]$;

$$c \le \left(1 - \frac{K}{\mu_t}\right)\pi + \frac{\|f_t\|_p^2}{\mu_t}\left(\frac{(b-a)^{q+2} - (b-K)^{q+2} - (K-a)^{q+2}}{(q+1)(q+2)}\right)^{1/q} \mathbf{1}(a < K < b)$$

provided that $f_t \in L_p[a,b]$, $p > 1$, $1/p + 1/q = 1$; and

$$c \le \frac{(\mu_t - K)p + (b-a)\,\mathbf{1}(a < K < b)}{\mu_t}.$$

These results can be applied, for example, in the problem of testing an option-pricing model's assumptions concerning the underlying asset's distribution using observed call option prices and in recovering the underlying asset's distributional characteristics from observed prices on derivatives on the asset.

3. Bounds on the expected payoffs of options in the binomial model

Consider a multiperiod binomial model on an asset with the price process $S_0 = s$ and $S_t = S_{t-1}X_t(p_t)$ for $t \ge 1$, where $X_t(p_t)$ are independent random variables with distributions $\mathrm{P}(X_t(p_t) = u) = p_t \in (0,1)$ and $\mathrm{P}(X_t(p_t) = d) = 1 - p_t$ for $0 < d < u$. The following theorem gives sharp estimates for the time-t expected payoffs of a European call option with strike price K on the asset expiring at time T.

Denote by $\bar{p}_{t,T}$ the average of the probabilities p_i for $i = t+1, \dots, T$, so $\bar{p}_{t,T} = \sum_{i=t+1}^{T} p_i/(T-t)$, and by $\theta_{t,T}$ a Poisson random variable with parameter $\sum_{i=t+1}^{T} p_i$. Below, $S_T(p_{t+1}, \dots, p_T) = S_t X_{t+1}(p_{t+1}) \cdots X_T(p_T)$ is the asset price at the call's expiration date.

Theorem 4. *The following sharp inequalities hold:*

$$\begin{aligned}\mathrm{E}_t \max[S_T(p_{t+1}, \dots, p_T) - K, 0] &\le \mathrm{E}_t \max[S_T(\bar{p}_{t+1,T}, \dots, \bar{p}_{t+1,T}) - K, 0]\\ &\le \mathrm{E}_t \max[S_t d^{T-t}(u/d)^{\theta_{t,T}} - K, 0].\end{aligned}$$

Remark 4. According to Theorem 4, the expected payoff of a call option in the multiperiod binomial model does not exceed the expected payoff of a call option in the two-period model written on an asset with the terminal price distributed as $S_t d^{T-t}(u/d)^{\theta_{t,T}}$. Thereby, the theorem allows us to reduce the problems of pricing options in a multiperiod setup to a two-period case.

Remark 5. From the proof of Theorem 4 it follows that similar results hold for expectations of any contingent claims with convex payoff functions. In particular, for power options with the payoff function $\max[(S_T - K)^p, 0]$, $p \ge 1$,

$$\begin{aligned}\mathrm{E}_t \max[(S_T(p_{t+1}, \dots, p_T) - K)^p, 0] &\le \mathrm{E}_t \max[(S_T(\bar{p}_{t+1,T}, \dots, \bar{p}_{t+1,T}) - K)^p, 0]\\ &\le \mathrm{E}_t \max[(S_t d^{T-t}(u/d)^{\theta_{t,T}} - K)^p, 0].\end{aligned}$$

Proof of Theorem 4. We have

$$\begin{aligned}&\mathrm{E}_t \max[S_T(p_{t+1}, \dots, p_T) - K, 0]\\ &\quad = \mathrm{E}_t \max\left[S_t d^{T-t} \exp\left((\log u - \log d) \sum_{i=t+1}^{T} Z_i(p_i)\right) - K, 0\right],\end{aligned}$$

where $Z_i(p_i)$ for $i = t+1, \dots, T$ are independent random variables with Bernoulli distributions $\mathrm{P}(Z_i(p_i) = 1) = p_i$ and $\mathrm{P}(Z_i(p_i) = 0) = 1 - p_i$ for $i = t+1, \dots, T$. According to Hoeffding (1956),

$$\mathrm{E}\,\phi\left(\sum_{i=t+1}^{T} Z_i(p_i)\right) \le \mathrm{E}\,\phi\left(\sum_{i=t+1}^{T} Z_i(\bar{p}_{t,T})\right) \tag{15}$$

for all convex functions $\phi : [0, \infty) \to \mathbb{R}$. Since the function

$$\tilde{\phi}(x) = \max[S_t d^{T-t} \exp((\log u - \log d)x) - K, 0]$$

is convex, we get from (15) the first inequality in Theorem 4. Furthermore, from (15) we obtain that

$$\mathrm{E}_t\, \tilde{\phi}\left(\sum_{i=t+1}^{T+k} Z_i\left(\frac{\sum_{i=t+1}^{T} p_i}{T-t+k}\right)\right) \le \mathrm{E}_t\, \tilde{\phi}\left(\sum_{i=t+1}^{T+k+1} Z_i\left(\frac{\sum_{i=t+1}^{T} p_i}{T-t+k+1}\right)\right) \tag{16}$$

for $k \ge 0$. We have

$$\begin{aligned}
\mathrm{E}_t\, \tilde{\phi}^2\left(\sum_{i=t+1}^{T+k} Z_i\left(\frac{\sum_{i=t+1}^{T} p_i}{T-t+k}\right)\right) &\le S_t^2 d^{2(T-t)}\, \mathrm{E}_t \exp\left[2(\log u - \log d)\sum_{i=t+1}^{T+k} Z_i\left(\frac{\sum_{i=t+1}^{T} p_i}{T-t+k}\right)\right] \\
&= S_t^2 d^{2(T-t)}\left[1 + \left(\left(\frac{u}{d}\right)^2 - 1\right)\frac{\sum_{i=t+1}^{T} p_i}{T-t+k}\right]^{T-t+k} \\
&\le S_t^2 d^{2(T-t)} \exp\left[\left(\left(\frac{u}{d}\right)^2 - 1\right)\left(\sum_{i=t+1}^{T} p_i\right)\right]
\end{aligned}$$

for $k \ge 0$, so that the sequence

$$\sum_{i=t+1}^{T+k} Z_i\left(\frac{\sum_{i=t+1}^{T} p_i}{T-t+k}\right)$$

for $k \ge 0$ is uniformly integrable. Since

$$\tilde{\phi}\left(\sum_{i=t+1}^{T+k} Z_i\left(\frac{\sum_{i=t+1}^{T} p_i}{T-t+k}\right)\right)$$

converges to $\tilde{\phi}(\theta_{t,T})$ in distribution as $k \to \infty$, from (15) and (16) we obtain the second inequality in Theorem 4. The proof is complete.

Consider now a cash-or-nothing option on the asset that pays an amount q if the terminal asset price S_T exceeds the exercise price K, that is, consider an option with the payoff function $q\,\mathbf{1}(S_T \ge K)$. The following theorem, which looks similar to the results of Eaton (1974) for the tail probabilities of weighted sums of independent symmetric Bernoulli random variables, gives bounds on the day-t expected payoff $\mathrm{E}_t\, q\,\mathbf{1}(S_T \ge K) = q\,\mathrm{P}_t(S_T \ge K)$ of the above option. Obviously, only the case where $K > S_t d^{T-t}$ is of interest since otherwise $\mathrm{E}_t\, q\,\mathbf{1}(S_T \ge K) = q$.

Theorem 5. *Let $K > S_t d^{T-t}$. Then the following estimate holds:*

$$q\,\mathrm{P}_t(S_T \ge K) \le q \inf_{0 \le u < v} \int_u^{\infty} \frac{x-u}{v-u}\, \mathrm{d}F_{\theta_{t,T}}, \tag{17}$$

where $v = \log(K/S_t d^{T-t})/(\log u - \log d)$ and $F_{\theta_{t,T}}(x)$ denotes the distribution function of the random variable $\theta_{t,T}$.

Proof. It is easy to see that

$$q\,\mathrm{P}_t(S_T \ge K) = q\,\mathrm{P}_t\left(\sum_{i=t+1}^{T} Z_i(p_i) \ge v\right), \tag{18}$$

where, as in the proof of Theorem 4, $Z_i(p_i)$ for $i = t+1, \ldots, T$ are independent random variables with distributions $\mathrm{P}(Z_i(p_i) = 1) = p_i$ and $\mathrm{P}(Z_i(p_i) = 0) = 1 - p_i$. From

Chebyshev's inequality it follows that

$$\mathrm{P}_t\bigg(\sum_{i=t+1}^{T} Z_i(p_i) \geq v\bigg) \leq \frac{\mathrm{E}_t \max\big[\sum_{i=t+1}^{T} Z_i(p_i) - u, 0\big]}{v-u} \tag{19}$$

for all $u \in [0, v)$. Since the function $\psi(x) = \max(x-u, 0)$ is convex, similarly to the proof of Theorem 4 we obtain that

$$\mathrm{E}_t \max\Bigg[\sum_{i=t+1}^{T} Z_i(p_i) - u, 0\Bigg] \leq \mathrm{E}\max[\theta_{t,T} - u, 0] \tag{20}$$

for $u \in [0, v)$. The relations (18)–(20) imply (17) and the proof is complete.

Fix $K > S_t d^{T-t}$ and consider, similarly to Eaton (1974), the class $\boldsymbol{F}_K$ of functions ϕ satisfying the conditions

$$\phi(x) = \begin{cases} \displaystyle\int_0^x (x-u)\,\mathrm{d}F(u), & x \geq 0, \\ 0, & x < 0, \end{cases}$$

$$\phi(\log(K/S_t d^{T-t})/(\log u - \log d)) = 1$$

for a nonnegative bounded nondecreasing function $F(x)$ on $[0, +\infty)$ with $F(0) = 0$. Similarly to the proof of Theorem 5 we obtain that

$$q\,\mathrm{P}_t(S_T \geq K) \leq q\,\mathrm{E}\,\phi(\theta_{t,T}) \tag{21}$$

for all $K > S_t d^{T-t}$ and $\phi \in \boldsymbol{F}_K$. It is not difficult to show, as in Eaton (1974, Proposition 4), that the estimates given by (17) are the best among all estimates (21), that is,

$$\inf_{\phi \in \boldsymbol{F}_K} \mathrm{E}\,\phi(\theta_{t,T}) = \inf_{0 \leq u < v} \int_u^{\infty} \frac{x-u}{v-u}\,\mathrm{d}F_{\theta_{t,T}}(x).$$

4. Conclusion

In this paper, we derived a number of semi-parametric bounds on the expected payoffs and prices of options in terms of the underlying asset's distributional characteristics. We obtained sharp bounds on the expected payoff and price of the European call option on an asset with an absolutely continuous price process in terms of the price density's norms. We also derived bounds on European call option expected payoffs in terms of the first two moments of the underlying asset-price returns and the probability of the option finishing in the money. The bounds are easily computable and can be better than those obtained by Lo (1987) and Grundy (1991).

We obtained sharp bounds on expected payoffs in the multiperiod binomial model that allow us to reduce the problems of approximating the expected payoffs in the setup to the one-period case. There has been little work on time-varying moments within the framework of the binomial model. Our model extends the functionality of such an approach. Our bounds on the multiperiod binomial model allow the probability of an up move to change with time; thus one time-varying moment, such as stochastic volatility or time-varying risk premiums, can be accommodated. An interesting extension of our model would be to extend the structure so that both stochastic volatility and time-varying risk premiums could be accommodated.

The advantage of the semi-parametric bounds derived in this paper is that, in order to calculate their values and thereby approximate the option prices, it is only necessary to assume the values of a minimal set of parameters for the distribution of the underlying asset's price process, i.e. knowledge of the entire distribution is not necessary. Furthermore, using lower bounds on the discount rate, it is possible to apply the bounds and observed option prices to determine a restricted set of feasible values of the underlying asset's price-distribution parameters and therefore test different option-pricing model assumptions concerning the underlying asset's distribution.

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