DETECTION THRESHOLDS FOR THE β-MODEL ON SPARSE GRAPHS

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APPENDIX A: PROOF OF BINOMIAL LEMMA

Proof. (a)

(i) The upper and lower bounds follow from (Bollobás, 2001, Theorem 1.2) and (Bollobás, 2001, Theorem 1.5) respectively.

(ii) The upper bound follows from (Bollobás, 2001, Theorem 1.3). The lower bound follows from part (Bollobás, 2001, Theorem 1.6) if $n\min(p_n, 1-p_n) \gg \log^3 n$, and from Part (a, i) otherwise.

(b)

(i) For the upper bound, fixing $\delta > 0$ and setting

$$t_n' := \delta\sqrt{np_n(1-p_n)\log n}, \quad t_n := np_n + C_n\sqrt{np_n(1-p_n)\log n}$$

we have

$$\begin{aligned} &\mathbb{P}(X_n + Y_n = t_n) \\ \le &\mathbb{P}(Y_n > t_n') + \max_{r=0}^{t_n'} \mathbb{P}(X_n = t_n - r) \sum_{r=0}^{t_n'} \mathbb{P}(Y_n = r) \\ \le &\mathbb{P}(Y_n > t_n') + \max_{r=0}^{t_n'} \mathbb{P}(X_n = t_n - r). \end{aligned}$$

For bounding the first term above, note that $\lim_{n\to\infty}(t_n' - b_n p_n') = \infty$, and so $t_n' \ge 2b_n p_n'$ for all n large. This observation, along with an application of Bernstein's inequality gives

$$\mathbb{P}(Y_n > t_n') \le \exp\Big\{ -\frac{\frac{1}{2}(t_n' - b_n p_n')^2}{b_n p_n'(1-p_n') + \frac{t_n' - b_n p_n'}{3}} \Big\} \le e^{-\frac{3}{24}t_n'}.$$

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But then $t'_n \gg \log n$, and so we have

$$\frac{1}{\log n} \log \mathbb{P}(Y_n > t'_n) = -\infty. \tag{A.1}$$

Thus it suffices to control the second term. Since

$$t_n - t'_n = np_n + (C_n - \delta)\sqrt{np_n(1-p_n)\log n},$$

the bound then follows to note that

$$\max_{0 \le r \le t'_n} \mathbb{P}(X_n = t_n - r) = \mathbb{P}(X_n = t_n - t'_n)$$

where we have used the fact that $t_n - t'_n - np_n \to \infty$ and the observation that the binomial distribution is unimodal. Finally, using part (e) gives

$$\mathbb{P}(X_n = t_n - t'_n) = \frac{1}{\sqrt{np_n}} n^{-\frac{(C-\delta)^2}{2} + o(1)},$$

from which the result follows since $\delta > 0$ is arbitrary.

Turning towards the lower bound, note that

$$\begin{aligned} \mathbb{P}(X_n + Y_n = t_n) &\ge \sum_{r=0}^{t'_n} \mathbb{P}(X_n = t_n - r)\mathbb{P}(Y_n = r) \\ &\ge \mathbb{P}(X_n = t_n)\mathbb{P}(Y_n \le t'_n), \end{aligned}$$

as $\min_{r=0}^{t'_n} \mathbb{P}(X_n = t_n - r) = \mathbb{P}(X_n = t_n)$. Since $\mathbb{P}(Y_n \le t'_n)$ converges to 1 (from (A.1)), the result then follows on using part (e).

(ii) For the upper bound, we have

$$\mathbb{P}(X_n + Y_n \ge t_n) \le \mathbb{P}(X_n \ge t_n - t'_n) + \mathbb{P}(Y_n \ge t'_n),$$

from which one can ignore $\mathbb{P}(Y_n \ge t'_n)$ using (A.1). Using part (a, ii) gives

$$\mathbb{P}(X_n \ge t_n - t'_n) \le n^{-\frac{(C-\delta)^2}{2} + o(1)},$$

from which the result follows since $\delta > 0$ is arbitrary.
Also in this case the lower bound follows trivially, on noting that

$$\mathbb{P}(X_n + Y_n \ge t_n) \ge \mathbb{P}(X_n \ge t_n)$$

and using part (a).

■

APPENDIX B: TECHNICAL LEMMAS FOR LOWER BOUND

B.1. Proof of Lemma 6.6. We recall from the proof of Theorem 3.2**i.**, the definitions of $f(x) = \frac{e^x}{1+e^x}$ and

$$h(x) = 4\mu f(c_1x)f(c_2x) + \frac{(1-2\mu f(c_1x))(1-2\mu f(c_2x))}{1-\mu},$$

where $0 \le \mu \le 1$ and $c_1, c_2 > 0$. Then note that $\Upsilon = \prod_{j=1}^{5} \Upsilon_i$, where

$$\Upsilon_1 = \left(4f(2A)^2\right)^{\Sigma(S_1\cap S_2)} \left(\frac{1-\frac{\lambda}{n}f(2A)}{1-\frac{\lambda}{2n}}\right)^{2(\binom{Z}{2}-\Sigma(S_1\cap S_2))},$$

$$\begin{aligned}\Upsilon_2 &= (4f(A)f(2A))^{\Sigma(S_1\cap S_2, S_1\Delta S_2)} \\ &\quad\times \left(\frac{1-\frac{\lambda}{n}f(2A)}{1-\frac{\lambda}{2n}}\cdot\frac{1-\frac{\lambda}{n}f(A)}{1-\frac{\lambda}{2n}}\right)^{2Z(s-Z)-\Sigma(S_1\cap S_2, S_1\Delta S_2)},\end{aligned}$$

$$\Upsilon_3 = \left(4f(A)^2\right)^{\Sigma(S_1\cap S_2^c, S_1^c\cap S_2)} \left(\frac{1-\frac{\lambda}{n}f(A)}{1-\frac{\lambda}{2n}}\right)^{2((s-Z)^2-\Sigma(S_1\cap S_2^c, S_1^c\cap S_2))},$$

$$\begin{aligned}\Upsilon_4 &= (4f(2A)f(0))^{\Sigma(S_1\cap S_2^c)+\Sigma(S_1^c\cap S_2)} \\ &\quad\times \left(\frac{1-\frac{\lambda}{n}f(2A)}{1-\frac{\lambda}{2n}}\right)^{2\binom{s-Z}{2}-(\Sigma(S_1\cap S_2^c)+\Sigma(S_1^c\cap S_2))},\end{aligned}$$

$$\begin{aligned}\Upsilon_5 &= (4f(A)f(0))^{\Sigma(S_1\Delta S_2, (S_1\cup S_2)^c)} \\ &\quad\times \left(\frac{1-\frac{\lambda}{n}f(A)}{1-\frac{\lambda}{2n}}\right)^{2(s-Z)(n-2s+Z)-\Sigma(S_1\Delta S_2, (S_1\cup S_2)^c)}.\end{aligned}$$

As in the proof of Theorem 3.2**i.**, we have that for each realization of S_1 and S_2, $\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\Upsilon_i)$ is exactly of the form h, with $\mu = \lambda/2n$ and appropriate c_1, c_2. We note that the Υ_i's are independent of each other and $\mathbb{E}_{\boldsymbol{\beta}=0,\lambda}[\Upsilon_5] = 1$. It follows, by arguments exactly similar to those leading to (6.9), that there exists universal constants $C_1, C_2 > 0$ such that

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\Upsilon) &\le \left(1 + C_1\frac{\lambda A^2}{n}\right)^{C_2s^2} \le \exp\left(C_1C_2\frac{s^2\lambda A^2}{n}\right) \\ &= \exp(C_1C_2C^*n^{1-2\alpha}\log n) = 1 + o(1), \quad \text{since} \quad \alpha > \frac{1}{2}.\end{aligned}$$

B.2. Proof of Lemma 6.7. The proof of the lemma follows from simple algebra along with the facts that $\lambda \leq n$, $s \ll \sqrt{n}$, and $\frac{1-\lambda/2n}{1-\theta} = 1 + o(1)$.

APPENDIX C: PROOF OF PROPOSITION 6.5

C.1. Proof of (6.12). We set

$$a(t) = \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(D_1 > t\right)$$

and

$$b(t) = \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(D_1 > t, D_2 > t\right)$$

to get

$$\mathrm{Var}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(HC(t)) = na(t)(1-a(t)) + n(n-1)(b(t) - a^2(t)).$$

D_1 and D_2 have some dependence through the common edge Y_{12}. We decompose the probabilities according to the value attained by Y_{12} and use the independence of the edges to get

$$\begin{aligned} b(t) &= \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(\frac{d_1 - \frac{\lambda}{2n}(n-1)}{\sqrt{(n-1)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}} > t, \frac{d_2 - \frac{\lambda}{2n}(n-1)}{\sqrt{(n-1)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}} > t\right) \\ &= \frac{\lambda}{2n}(a'(t))^2 + \left(1 - \frac{\lambda}{2n}\right)(a''(t))^2, \qquad \text{(C.1)} \end{aligned}$$

$$a'(t) = \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(\frac{\sum_{j\neq 2} Y_{1,j} - \frac{\lambda}{2n}(n-2)}{\sqrt{(n-2)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}} > t\sqrt{\frac{n-1}{n-2}} - \sqrt{\frac{1-\frac{\lambda}{2n}}{(n-2)\frac{\lambda}{2n}}}\right), \qquad \text{(C.2)}$$

$$a''(t) = \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(\frac{\sum_{j\neq 2} Y_{1,j} - \frac{\lambda}{2n}(n-2)}{\sqrt{(n-2)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}} > t\sqrt{\frac{n-1}{n-2}} + \sqrt{\frac{\frac{\lambda}{2n}}{(n-2)(1-\frac{\lambda}{2n})}}\right). \qquad \text{(C.3)}$$

Similarly, we condition on the value of Y_{12} and use the independence of edges to get

$$a(t) = \frac{\lambda}{2n}a'(t) + \left(1 - \frac{\lambda}{2n}\right)a''(t). \qquad \text{(C.4)}$$

Therefore, we have, using (C.1) and (C.4),

$$n(n-1)(b(t) - a^2(t)) = n(n-1)\frac{\lambda}{2n}\left(1 - \frac{\lambda}{2n}\right)(a'(t) - a''(t))^2. \qquad \text{(C.5)}$$

Now, using (C.2) and (C.3), we have,

$$a'(t) - a''(t) = \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(\sum_{j\neq 2} Y_{1,j} = \left\lceil (n-1)\frac{\lambda}{2n} + t\sqrt{(n-1)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}\right\rceil - 1\right)$$
$$= \frac{n^{-r+o(1)}}{\sqrt{\lambda}},$$

where the last line uses Part (a, i) of Lemma 6.2, along with the fact that $\sum_{j\neq 2} Y_{1,j} \sim \mathrm{Bin}(n-2, p)$ with $p = \lambda/2n$.

Therefore, we have, using (C.5),

$$n(n-1)(b(t) - a^2(t)) \leq n(n-1)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)\frac{n^{-2r+o(1)}}{\lambda} = O(n^{1-2r+o(1)}). \tag{C.6}$$

Also by Lemma 6.2 Part (a, ii),

$$a(t) = \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(D_1 > t) = n^{-(r+o(1))}. \tag{C.7}$$

Therefore, combining (C.6) and (C.7) we get

$$\begin{aligned}\mathrm{Var}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(HC(t)) &= na(t)(1-a(t)) + n(n-1)(b(t) - a^2(t)) \\ &= n^{1-r+o(1)}.\end{aligned} \tag{C.8}$$

This completes the proof of (6.12).

C.2. Proof of (6.14)**.** Recall that the alternative distribution $\mathbb{P}_{\boldsymbol{\beta},\lambda}$ is such that $\beta_i = A$ for $i \in S$ and $\beta_i = 0$ otherwise, where $A = \sqrt{C^* \frac{\log n}{\lambda}}$ with $16(1-\theta) \geq C^* > C_{\mathrm{sparse}}(\alpha)$, $\theta = \lim \frac{\lambda}{2n}$, $|S| = s = n^{1-\alpha}$, $\alpha \in (1/2, 1)$. We begin with the following set of notation.

$$\begin{aligned}
a^{(s)}(t) &= \mathbb{P}_{\boldsymbol{\beta},\lambda}(D_i > t), \quad i \in S, \\
a^{(n-s)}(t) &= \mathbb{P}_{\boldsymbol{\beta},\lambda}(D_i > t), \quad i \in S^c, \\
b^{(s)}(t) &= \mathbb{P}_{\boldsymbol{\beta},\lambda}(D_i > t, D_j > t), \quad (i,j) \in S \times S, \\
b^{(n-s)}(t) &= \mathbb{P}_{\boldsymbol{\beta},\lambda}(D_i > t, D_j > t), \quad (i,j) \in S^c \times S^c, \\
b^{(s,n-s)}(t) &= \mathbb{P}_{\boldsymbol{\beta},\lambda}(D_i > t, D_j > t), \quad (i,j) \in S \times S^c,
\end{aligned} \tag{C.9}$$

The variance of $HC(t)$ under any $\mathbb{P}_{\boldsymbol{\beta},\lambda}$ considered above can be decomposed as follows.

$$\mathrm{Var}_{\boldsymbol{\beta},\lambda}(HC(t)) := \sum_{i=1}^{5} T_i, \tag{C.10}$$

$$T_1 = sa^{(s)}(t)(1-a^{(s)}(t)),$$
$$T_2 = (n-s)a^{(n-s)}(t)(1-a^{(n-s)}(t)),$$
$$T_3 = s(s-1)(b^{(s)}(t)-(a^{(s)}(t))^2),$$
$$T_4 = (n-s)(n-s-1)(b^{(n-s)}(t)-(a^{(n-s)}(t))^2),$$
$$T_5 = 2s(n-s)(b^{(s,n-s)}(t)-a^{(s)}(t)a^{(n-s)}(t)).$$

The basic idea of the proof is that the diagonal terms T_1, T_2 dominate over the covariance terms T_3, T_4 and T_5. The next Lemma collects the necessary details.

LEMMA C.1. *Fix* $\theta = \lim_{n\to\infty} \frac{\lambda}{2n}$. *For* $t = \lfloor\sqrt{2r\log n}\rfloor$ *with* $r > \frac{C^*}{16(1-\theta)}$, *we have,*

$$\lim_{n\to\infty}\frac{\log T_1}{\log n} = 1-\alpha-\frac{1}{2}\left(\sqrt{2r}-\sqrt{\frac{C^*}{8(1-\theta)}}\right)^2, \quad \lim_{n\to\infty}\frac{\log T_2}{\log n} = 1-r,$$
$$\lim_{n\to\infty}\frac{\log T_3}{\log n} \le 1-2\alpha-\left(\sqrt{2r}-\sqrt{\frac{C^*}{8(1-\theta)}}\right)^2, \quad \lim_{n\to\infty}\frac{\log T_4}{\log n} \le 1-2r,$$
$$\lim_{n\to\infty}\frac{\log T_5}{\log n} \le 1-\alpha-\frac{1}{2}\left(\sqrt{2r}-\sqrt{\frac{C^*}{8(1-\theta)}}\right)^2-r.$$

Lemma C.1 along with (C.10) immediately implies (6.14). We outline the proof of Lemma C.1 in the rest of the section.

PROOF OF LEMMA C.1. We begin by proving the bound on T_3 which is the most involved and captures the idea behind the asymptotic behavior of the other terms as well. Throughout this proof, we set $f(x) = e^x/(1+e^x)$. Using a conditioning argument as in (C.1), we have, for a pair $(i,j) \in S\times S$,

$$b^{(s)}(t) = \frac{\lambda}{n}f(2A)(a^{(s)'}(t))^2 + \left(1-\frac{\lambda}{n}f(2A)\right)(a^{(s)''}(t))^2,$$
$$a^{(s)'}(t) = \mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\frac{\sum_{l\neq j}Y_{i,l}-\frac{\lambda}{2n}(n-2)}{\sqrt{(n-2)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}} > t\sqrt{\frac{n-1}{n-2}}-\sqrt{\frac{1-\frac{\lambda}{2n}}{(n-2)\frac{\lambda}{2n}}}\right), \tag{C.11}$$
$$a^{(s)''}(t) = \mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\frac{\sum_{l\neq j}Y_{i,l}-\frac{\lambda}{2n}(n-2)}{\sqrt{(n-2)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}} > t\sqrt{\frac{n-1}{n-2}}+\sqrt{\frac{\frac{\lambda}{2n}}{(n-2)(1-\frac{\lambda}{2n})}}\right). \tag{C.12}$$

Using the same argument as (C.4), we have,

$$a^{(s)}(t) = \frac{\lambda}{n} f(2A) a^{(s)'}(t) + \left(1 - \frac{\lambda}{n} f(2A)\right) a^{(s)''}(t).$$

Therefore, we have, using (C.10),

$$T_3 = s(s-1)\frac{\lambda}{n} f(2A)\left(1 - \frac{\lambda}{n} f(2A)\right)(a^{(s)'}(t) - a^{(s)''}(t))^2. \tag{C.13}$$

Combining (C.11) and (C.12), we have,

$$a^{(s)'}(t) - a^{(s)''}(t) = \mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\sum_{l\neq j} Y_{i,l} = \left\lfloor (n-1)\frac{\lambda}{2n} + t\sqrt{(n-1)\frac{\lambda}{2n}\left(1 - \frac{\lambda}{2n}\right)}\right\rfloor\right).$$

We note that $\sum_{l\neq j} Y_{i,l} \overset{d}{=} Z_1 + Z_2$ where $Z_1 \sim \mathrm{Bin}\left(s-2, \frac{\lambda}{n} f(2A)\right)$, $Z_2 \sim \mathrm{Bin}\left(n-s, \frac{\lambda}{n} f(A)\right)$, and Z_1, Z_2 are independent random variables. Setting $t_n = \left\lfloor (n-1)\frac{\lambda}{2n} + t\sqrt{(n-1)\frac{\lambda}{2n}\left(1 - \frac{\lambda}{2n}\right)}\right\rfloor$, by direct computation, we have,

$$t_n = (n-s)\frac{\lambda}{n} f(A) + C_n\sqrt{(n-s)\frac{\lambda}{n} f(A)\left(1 - \frac{\lambda}{n} f(A)\right)\log(n-s)},$$

such that $C_n \to \sqrt{2r} - \sqrt{\frac{C^*}{8(1-\theta)}} > 0$ as $n \to \infty$. Therefore, applying Lemma 6.2 Part (b, i), we have,

$$a^{(s)'}(t) - a^{(s)''}(t) \le \frac{n^{-(\sqrt{r} - \sqrt{C^*/16(1-\theta)})^2 + o(1)}}{\sqrt{\lambda}}. \tag{C.14}$$

Plugging this bound back into (C.13) immediately implies that

$$T_3 \le n^{1 - 2\alpha - \left(\sqrt{2r} - \sqrt{C^*/(8(1-\theta))}\right)^2 + o(1)}.$$

This completes the bound on T_3.

We next turn to the bound on T_4. We use the same argument in this case to derive the following expression for T_4 which is exactly comparable to (C.13). We have,

$$T_4 = (n-s)(n-s+1)\frac{\lambda}{n} f(0)\left(1 - \frac{\lambda}{n} f(0)\right)(a^{(n-s)'}(t) - a^{(n-s)''}(t))^2, \tag{C.15}$$

where $a^{(n-s)'}(t)$, $a^{(n-s)''}(t)$ are defined analogous to $a^{(s)'}(t)$ and $a^{(s)''}(t)$ in (C.11), (C.12) respectively. The same argument as in the bound on T_3 now implies that

$$T_4 \leq n^{1-2r+o(1)}.$$

This gives us the desired bound on T_4.

Finally, we control the last covariance term T_5. In this case, we fix $(i,j) \in S \times S^c$. Recall the notations introduced in (C.9). We have, similar to (C.1),

$$b^{(s,n-s)}(t) = \frac{\lambda}{n} f(A) a^{(s,n-s)'}(t) a^{(n-s,s)'}(t) + \left(1 - \frac{\lambda}{n} f(A)\right) a^{(s,n-s)''}(t) a^{(n-s,s)''}(t),$$

$$a^{(s,n-s)'}(t) = \mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\frac{\sum_{l\neq j} Y_{i,l} - \frac{\lambda}{2n}(n-2)}{\sqrt{(n-2)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}} > t\sqrt{\frac{n-1}{n-2}} - \sqrt{\frac{1-\frac{\lambda}{2n}}{(n-2)\frac{\lambda}{2n}}}\right), \tag{C.16}$$

$$a^{(s,n-s)''}(t) = \mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\frac{\sum_{l\neq j} Y_{i,l} - \frac{\lambda}{2n}(n-2)}{\sqrt{(n-2)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}} > t\sqrt{\frac{n-1}{n-2}} + \sqrt{\frac{\frac{\lambda}{2n}}{(n-2)(1-\frac{\lambda}{2n})}}\right). \tag{C.17}$$

$a^{(n-s,s)'}(t)$ and $a^{(n-s,s)''}(t)$ are defined by switching the roles of i,j in (C.16) and (C.17) respectively.

Similar to (C.4), we have,

$$a^{(s)}(t) = \frac{\lambda}{n} f(A) a^{(s,n-s)'}(t) + \left(1 - \frac{\lambda}{n} f(A)\right) a^{(s,n-s)''}(t),$$
$$a^{(n-s)}(t) = \frac{\lambda}{n} f(A) a^{(n-s,s)'}(t) + \left(1 - \frac{\lambda}{n} f(A)\right) a^{(n-s,s)''}(t).$$

Therefore, we have,

$$\begin{aligned} T_5 = 2s(n-s)\frac{\lambda}{n} f(A)\left(1 - \frac{\lambda}{n} f(A)\right)\left(a^{(s,n-s)'}(t) - a^{(s,n-s)''}(t)\right) \\ \times \left(a^{(n-s,s)'}(t) - a^{(n-s,s)''}(t)\right). \end{aligned} \tag{C.18}$$

We bound $\left(a^{(s,n-s)'}(t) - a^{(s,n-s)''}(t)\right)$ and $\left(a^{(n-s,s)'}(t) - a^{(n-s,s)''}(t)\right)$ exactly as (C.14) to obtain

$$\left(a^{(s,n-s)'}(t) - a^{(s,n-s)''}(t)\right) \leq \frac{n^{-(\sqrt{r}-\sqrt{C^*/16(1-\theta)})^2+o(1)}}{\sqrt{\lambda}},$$

$$\left(a^{(n-s,s)'}(t) - a^{(n-s,s)''}(t)\right) \leq \frac{n^{-r+o(1)}}{\sqrt{\lambda}}.$$

Plugging these bounds back into (C.18) completes the proof.

It remains to study the diagonal terms T_1, T_2. Recall that for $i \in S$,

$$\begin{aligned}
&a^{(s)}(t) \\
&= \mathbb{P}_{\boldsymbol{\beta},\lambda}\left(d_i > (n-1)\frac{\lambda}{2n} + t\sqrt{(n-1)\frac{\lambda}{2n}\left(1-\frac{\lambda}{2n}\right)}\right), \\
&= \mathbb{P}_{\boldsymbol{\beta},\lambda}\left(d_i > (n-s)\frac{\lambda}{n}f(A) + C_n\sqrt{(n-s)\frac{\lambda}{n}f(A)\left(1-\frac{\lambda}{n}f(A)\right)\log(n-s)}\right).
\end{aligned}$$

for a sequence $C_n \to \sqrt{2r} - \sqrt{\frac{C^*}{8(1-\theta)}} > 0$. We note that $d_i = Z_1 + Z_2$, where $Z_1 \sim \text{Bin}(s, \frac{\lambda}{n}f(2A))$ and $Z_2 \sim \text{Bin}(n-2, \frac{\lambda}{n}f(A))$ are independent random variables. Thus using Lemma 6.2 part (b, ii), we have,

$$a^{(s)}(t) = n^{-\frac{1}{2}(\sqrt{2r}-\sqrt{\frac{C^*}{8(1-\theta)}})^2+o(1)}.$$

Plugging this bound back into the definition of T_1 gives us the desired result. The proof for T_2 is exactly similar to that of T_4 and is therefore omitted. ∎

C.3. Proof of (6.13)**.** Recall the definition of $a(t)$ from the proof of (6.12). In this case, we have, for $i \in S, j \in S^c$,

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\beta},\lambda}\left(HC(t)\right) &= s\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(D_i > t\right) + (n-s)\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(D_j > t\right) - na(t) \\
&\geq s\left(\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(D_i > t\right) - \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(D_i > t\right)\right),
\end{aligned}$$

using the fact that the vertex degrees are stochastically larger under the alternative. An application of Lemma 6.2 Part (a, ii) and (b, ii), implies that for $r > \frac{C^*}{16(1-\theta)}$

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\beta},\lambda}\left(HC(t)\right) &\geq s\left(n^{-\left(\sqrt{2r}-\sqrt{C^*/8(1-\theta)}\right)^2/2+o(1)} - n^{-r+o(1)}\right) \\
&\geq n^{1-\alpha-\left(\sqrt{2r}-\sqrt{C^*/8(1-\theta)}\right)^2/2+o(1)}.
\end{aligned}$$

This completes the proof.

APPENDIX D: PROOF OF THEOREM 3.3

Proof of 3.3**i.**. Using monotonicity arguments without loss of generality one can consider the alternative $\mathbb{P}_{\boldsymbol{\beta},\lambda}$ where β is given by

$$\beta_i = A \text{ for } i \in S, \quad 0 \text{ otherwise},$$

where $A = \sqrt{C^* \frac{\log n}{\lambda}}$ for some C^* with

$$16(1-\theta) \geq C^* > C_{\max}(\alpha) := 16(1-\theta)(1-\sqrt{1-\alpha})^2,$$

and $|S| = s = n^{1-\alpha}$. Given C^* let $\delta > 0$ be such that

$$C^* > 16(1-\theta)[\sqrt{1+\delta} - \sqrt{1-\alpha}]^2.$$

and let ϕ_n be the sequence of tests which rejects when $\max_{i\in[n]} d_i > k_n(\delta)$, and accepts otherwise, where

$$\begin{aligned} p_n &:= \frac{\lambda}{2n}. \\ k_n(\delta) &:= np_n + \sqrt{2(1+\delta)np_n(1-p_n)\log n} \\ &= \frac{\lambda}{2} + \sqrt{(1+\delta)\lambda\Big(1 - \frac{\lambda}{2n}\Big)\log n}. \end{aligned}$$

Thus, using FKG inequality gives

$$1 - \mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\phi_n \geq \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(d_1 \leq k_n(\delta))^n \geq \Big(1 - n^{-(1+\delta+o(1))}\Big)^n,$$

where the last inequality uses Part (a, ii) of Lemma 6.2. Since the RHS above converges to 1, it is enough to show that

$$\sup_{\beta\in\Xi(s,A)} \mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i=1}^n d_i \leq k_n(\delta)) \overset{n\to\infty}{\to} 0.$$

With $d_1', \cdots, d_s' \overset{i.i.d.}{\sim} \text{Bin}(n-s, p_n')$ with $p_n' := \frac{\lambda}{n}\frac{e^A}{1+e^A}$, it is easy to see that $\max_{i=1}^n d_i$ is stochastically larger than $\max_{i=1}^s d_i'$, and so it suffices to show that

(D.1)
$$\mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i=1}^s d_i' \leq k_n(\delta)) = \mathbb{P}(d_1' \leq k_n(\delta))^s = (1 - \mathbb{P}(d_1' > k_n(\delta)))^s \overset{n\to\infty}{\to} 0.$$

To this effect, note that

$$\lim_{n\to\infty} \frac{k_n(\delta) - (n-s)p_n'}{\sqrt{(n-s)p_n'(1-p_n')\log(n-s)}} = -\frac{\sqrt{2C^*}}{4\sqrt{1-\theta}} + \sqrt{2(1+\delta)}.$$

Also note that the assumption $C^* \le 16(1-\theta)$ implies the limit above is positive. Since $s = n^{1-\alpha}$ and $(1-x) = \exp(-x + O(x^2))$ as $x \to 0$, (D.1) will follow from Lemma 6.2 Part (a, ii) if we can show that

$$\Big[-\frac{\sqrt{2C^*}}{4\sqrt{1-\theta}} + \sqrt{2(1+\delta)}\Big]^2 < 2(1-\alpha)$$
$$\Leftrightarrow -\frac{\sqrt{C^*}}{4\sqrt{1-\theta}} + \sqrt{1+\delta} < \sqrt{1-\alpha}$$
$$\Leftrightarrow \frac{\sqrt{C^*}}{4\sqrt{1-\theta}} > \sqrt{1+\delta} - \sqrt{1-\alpha},$$

which holds by choice of δ. This completes the proof of the upper bound. ∎

Proof of ii.. To show the lower bound, again consider an alternative of the form $\mathbb{P}_{\beta,\lambda}$, where β is given by

$$\beta_i = A \text{ for } i \in S, \quad 0 \text{ otherwise},$$

where $A = \sqrt{C^* \frac{\log n}{\lambda}}$ for some positive C^* with

$$C^* < C_{\max}(\alpha) := 16(1-\theta)(1-\sqrt{1-\alpha})^2.$$

and $|S| = s = n^{1-\alpha}$.

Suppose, to the contrary, that there is a sequence of consistent tests based on $\max_{i\in[n]} d_i$. Thus there exists sequence of positive reals $\{k_n\}_{n\ge 1}$ such that

$$\lim_{n\to\infty} \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\max_{i\in[n]} d_i \le k_n) = 1, \quad \lim_{n\to\infty} \mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i\in[n]} d_i \le k_n) = 0.$$

Denote $p_n = \lambda/2n$, $q_n = 1 - p_n$ and consider the sequence δ_n such that

$$k_n = np_n + (2np_nq_n \log n)^{1/2}\left(1 - \frac{\log\log n + \log(4\pi)}{4\log n} + \frac{\delta_n}{2\log n}\right).$$

We first claim that $\delta_n \to \infty$. The proof of the claim follows immediately since by Corollary 3.4 of Bollobás (2001), we have for any fixed $\delta \in \mathbb{R}$ that

$$\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(\max_{i=1}^{n} d_i < np_n + (2np_nq_n\log n)^{1/2}\left(1 - \frac{\log\log n + \log(4\pi)}{4\log n} + \frac{\delta}{2\log n}\right)\right)$$
$$\to e^{-e^{-\delta}}, \tag{D.2}$$

whenever $\frac{np_nq_n}{(\log n)^3} \to \infty$, a condition that holds by our assumption of $\lambda \gg (\log n)^3$. Indeed, if $\delta_n \le M$ for some $0 < M < \infty$, then

$$\limsup \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\max_{i=1}^{n} d_i \le k_n) \le e^{e^{-M}} < 1.$$

We now show that for any such choice of k_n the probability of making a type II error converges to 1. To this end note that letting $S:=\{l\colon \beta_l \neq 0\}$, by union bound we have the following inequality for $i \in S$

$$\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\max_{i=1}^{n} d_i > k_n\right) \leq s\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(d_i > k_n\right) + \mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\max_{j\in S^c} d_j > k_n\right).$$

We now show that individually, the two summands in the last display converges to 0. First, if $i \in S:=\{l\colon \beta_l \geq 0\}$, we have for $X_n' \perp Y_n'$ with $X_n' \sim \mathrm{Bin}\left(s-1, \frac{\lambda}{n}f(2A)\right)$, $Y_n' \sim \mathrm{Bin}\left(n-s, \frac{\lambda}{n}f(A)\right)$ (where as usual $f(x) = \frac{e^x}{1+e^x}$)

$$s\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(d_i > k_n\right) = s\mathbb{P}_{\boldsymbol{\beta},\lambda}(X_n' + Y_n' > k_n) \leq n^{1-\alpha-\left(1-\sqrt{C^*/16(1-\theta)}\right)^2+o(1)} = o(1).$$

The last inequality follows by calculations similar to those leading to proof of Lemma 6.5 upon invoking Lemma 6.2 Part (b, ii) and the last equality follows by the property of $C^* < C_{\max}(\alpha)$.

The control of $\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\max_{j\in S^c} d_j > k_n\right)$ is in philosophy similar to that of understanding the null behavior of $\max_{i=1}^{n} d_i$. However, one needs to carefully overcome the contamination by signals in each of the degrees involved. In particular, for any sequence k_n',

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\max_{j\in S^c} d_j > k_n\right) &= \mathbb{P}_{\beta}\left(\max_{j\in S^c}(X_j + Y_j) > k_n\right) \\
&\leq \mathbb{P}_{\beta}\left(\max_{j\in S^c} X_j + \max_{j\in S^c} Y_j > k_n\right) \\
&\leq \mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i=1}^{n} X_i > k_n') + \mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i=1}^{n} Y_i > k_n - k_n'),
\end{aligned} \tag{D.3}$$

where $X_i \overset{i.i.d.}{\sim} \mathrm{Bin}(s, \frac{\lambda}{n}f(A))$ and $Y_i \overset{i.i.d.}{\sim} \mathrm{Bin}(n-s-1, \frac{\lambda}{n}f(0))$. We choose $k_n' = \frac{\sqrt{np_nq_n}a_n}{\sqrt{2\log n}}$ for some sequence $a_n \to \infty$ sufficiently slow, to be chosen appropriately. Then by union bound and Bernstein's Inequality

$$\begin{aligned}
&\mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i=1}^{n} X_i > k_n') \\
&\leq n\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(X_1 - s\frac{\lambda}{n}f(A) > k_n' - s\frac{\lambda}{n}f(A)\right) \\
&\leq n\exp\left(-\frac{\frac{1}{2}(k_n'')^2}{s\frac{\lambda}{n}f(A)(1-\frac{\lambda}{n}f(A)) + \frac{1}{3}k_n''}\right), \quad k_n'' = k_n' - s\frac{\lambda}{n}f(A)
\end{aligned}$$

$$\leq n \exp\left(-Ck_n'\right),$$

where the last inequality holds for a universal constant $C > 0$ since for any $a_n \to \infty$, $k_n' \gg s\frac{\lambda}{n}f(A)$ (since $s = n^{1-\alpha}$, $\alpha > \frac{1}{2}$, and $\lambda \leq n$). Since $\lambda \gg (\log n)^3$ we have $k_n' \gg \log n$ and therefore

$$\mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i=1}^{n} X_i > k_n') \to 0, \tag{D.4}$$

as $n \to \infty$. Now note that by stochastic ordering

$$\mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i=1}^{n} Y_i > k_n - k_n') \leq \mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i=1}^{n} Y_i' > k_n - k_n')$$

where $Y_i' \overset{i.i.d.}{\sim} \text{Bin}(n, p_n)$. But,

$$\begin{aligned} &k_n - k_n' \\ &= np_n + (2np_nq_n \log n)^{1/2}\left(1 - \frac{\log\log n}{4\log n} - \frac{\log(2\pi^{1/2})}{2\log n} + \frac{\delta_n - a_n}{2\log n}\right) \end{aligned}$$

For any sequence $\delta_n \to \infty$, we can choose $a_n \to \infty$ sufficiently slow such that $\delta_n - a_n \to \infty$ as $n \to \infty$. But due to convergence to a continuous distribution, the convergence in (D.2) is uniform. Therefore

$$\mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i=1}^{n} Y_i' > k_n - k_n') \to 0. \tag{D.5}$$

The proof is therefore complete by combining (D.3), (D.4), and (D.5). ∎

APPENDIX E: ALTERNATE LOWER BOUND ARGUMENT FOR MAXIMUM DEGREE TEST

We recall that lower bound in Theorem 3.3 contains a gap regarding $\log n \ll \lambda \lesssim \log^3 n$. In this section we show that for $\log n \ll \lambda \lesssim \log^3 n$, if one considers the Maximum Degree Test that rejects when $\max_{i=1}^{n} d_i > np_n + \sqrt{\delta_n np_nq_n \log n}$, where $p_n = \lambda/2n$, $q_n = 1 - p_n$, and δ_n is some sequence of real numbers, then such tests are asymptotically powerless as soon as $C^* < C_{\max}(\alpha)$ defined above, if $\limsup \delta_n \neq 2$. The case when $\limsup \delta_n = 2$ is extremely challenging, and the result of the testing problem depends on the rate of convergence of δ_n to 2 along subsequences.

In particular, once again consider an alternative of the form $\mathbb{P}_{\beta,\lambda}$, where β is given by

$$\beta_i = A \text{ for } i \in S, \quad 0 \text{ otherwise},$$

where $A = \sqrt{C^* \frac{\log n}{\lambda}}$ for some positive C^* with

$$C^* < C_{\max}(\alpha) := 16(1-\theta)(1-\sqrt{1-\alpha})^2.$$

and $|S| = s = n^{1-\alpha}$.

Suppose to the contrary, there exists sequence of positive reals $\{k_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\max_{i\in[n]} d_i \leq k_n) = 1, \quad \lim_{n\to\infty} \mathbb{P}_{\boldsymbol{\beta},\lambda}(\max_{i\in[n]} d_i \leq k_n) = 0.$$

Suppose $k_n = np_n + \delta_n \sqrt{\log n}\sqrt{np_nq_n}$, and let $\overline{\delta} = \limsup \delta_n$, $\underline{\delta} = \liminf \delta_n$.

Suppose $\overline{\delta} \leq 0$. Then arguing along a subsequence

$$\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\max_{i=1}^{n} d_i > k_n) \geq \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(d_1 \leq np_n) \geq \frac{1}{4}$$

by looking at the median of d_1 under $\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}$, if λ is even.

Therefore, we can safely assume that $\exists\ \delta > 0$ such that $\overline{\delta} \geq \delta$. Subsequently, we work along the subsequence along which δ_n eventually becomes at least as large as δ. Now if $i \in S := \{l : \beta_l \neq 0\}$, we have along an appropriate subsequence, for $X_n \perp Y_n$ with $X_n \sim \text{Bin}\left(s-1, \frac{\lambda}{n} f(2A)\right)$, $Y_n \sim \text{Bin}\left(n-s, \frac{\lambda}{n} f(A)\right)$ (where as usual $f(x) = \frac{e^x}{1+e^x}$)

$$\begin{aligned}
&\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(d_i \leq k_n\right) = \mathbb{P}_{\boldsymbol{\beta},\lambda}(X_n + Y_n \leq k_n) \\
&= 1 - \mathbb{P}_{\boldsymbol{\beta},\lambda}(X_n + Y_n > k_n) \\
&\geq 1 - \mathbb{P}_{\boldsymbol{\beta},\lambda}(X_n + Y_n > np_n + \delta\sqrt{\log n}\sqrt{np_nq_n}) \\
&\geq 1 - n^{-\left(\frac{\delta}{\sqrt{2}} - \sqrt{C^*/16(1-\theta)}\right)^2 + o(1)}.
\end{aligned}$$

The last inequality follows by calculations similar to those leading to the behavior of T_1 in Lemma 6.5 upon invoking Lemma 6.2 Part (b, ii).

Similarly if $i \in S^c$, we have for $X_n \perp Y_n$ with $X_n \sim \text{Bin}\left(s, \frac{\lambda}{n} f(A)\right)$, $Y_n \sim \text{Bin}\left(n-s-1, \frac{\lambda}{n} f(0)\right)$

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(d_i \leq k_n\right) &= \mathbb{P}_{\boldsymbol{\beta},\lambda}(X_n + Y_n \leq k_n) \\
&= 1 - \mathbb{P}_{\boldsymbol{\beta},\lambda}(X_n + Y_n > k_n) \\
&\geq 1 - \mathbb{P}_{\boldsymbol{\beta},\lambda}(X_n + Y_n > np_n + \delta\sqrt{\log n}\sqrt{np_nq_n}) \\
&\geq 1 - n^{-\frac{\delta^2}{2} + o(1)}.
\end{aligned}$$

Therefore, by FKG inequality

$$\mathbb{P}_{\boldsymbol{\beta},\lambda}\left(\max_{i=1}^{n} d_i \leq k_n\right) \geq \prod_{i=1}^{n} \mathbb{P}_{\boldsymbol{\beta},\lambda}\left(d_i \leq k_i\right)$$

$$\geq \exp\left((n-s)\log\left(1-n^{-\frac{\delta^2}{2}+o(1)}\right)+s\log\left(1-n^{-\left(\frac{\delta}{\sqrt{2}}-\sqrt{C^*/16(1-\theta)}\right)^2+o(1)}\right)\right)$$
$$\sim \exp\left(-n^{1-\frac{\delta^2}{2}+o(1)}(1+o(1))-n^{1-\alpha-\left(\frac{\delta}{\sqrt{2}}-\sqrt{C^*/16(1-\theta)}\right)^2+o(1)}(1+o(1))\right).$$

Now suppose that $\delta > \sqrt{2}$. Then $1-\delta^2/2 < 0$ and note that $1-\alpha-\left(\frac{\delta}{\sqrt{2}}-\sqrt{C^*/16(1-\theta)}\right)^2 < 0$ since the function

$$g(x) = 1-\alpha-\left(x-\sqrt{C^*/16(1-\theta)}\right)^2$$

is decreasing for for $x \geq \sqrt{C^*/16(1-\theta)} < 1$, and $g(1) < 0$. Therefore, $\delta \leq \sqrt{2}$. Let us consider the case when $\delta < \sqrt{2}$ first.

Then there exists $\varepsilon > 0$ such that $\delta = \sqrt{2(1-\varepsilon)}$. We now show that $\exists\ \eta > 0$ such that $\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\max_{i=1}^n d_i > k_n) \geq \eta$, and thereby arriving at a contradiction. To show this we use a second moment method in conjunction with Paley-Zygmund Inequality.

In particular, define $\zeta_i = \mathcal{I}(d_i > k_n), \quad i = 1,\ldots,n$. Then by Paley-Zygmund Inequality

$$\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(\sum_{i=1}^n \zeta_i > \frac{1}{2}\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^n \zeta_i)\right) \geq \frac{1}{4}\frac{(\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^n \zeta_i))^2}{\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^n \zeta_i)^2}. \tag{E.1}$$

Now with a proof similar to that of (6.13) in Lemma 6.5

$$\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^n \zeta_i) = n\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(d_1 > k_n) \gtrsim n^{1-(1-\varepsilon)^2+o(1)}. \tag{E.2}$$

Also by a proof similar to that of (C.6)

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^n \zeta_i)^2 \\
&= n\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(d_1 > k_n) + n(n-a)(\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(d_1 > k_n, d_2 > k_n) \\
&- \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(d_1 > k_n)\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(d_2 > k_n)) + n(n-1)\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(d_1 > k_n)\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(d_2 > k_n) \\
&\lesssim n^{1-(1-\varepsilon)^2+o(1)} + n^{1-2(1-\varepsilon)+o(1)} + n^{2-2(1-\varepsilon)^2+o(1)}.
\end{aligned} \tag{E.3}$$

Therefore, combining (E.4), (E.2), and (E.3), we have the existence of an $\eta > 0$ such that

$$\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}\left(\sum_{i=1}^n \zeta_i > \frac{1}{2}\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^n \zeta_i)\right) \geq \frac{4\eta}{4} = \eta. \tag{E.4}$$

This in turn implies the proof in the case of $\delta > \sqrt{2}$ since

$$\begin{aligned}\mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\max_{i=1}^{n} d_i > k_n) &= \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^{n} \zeta_i \geq 1) \\ &\geq \mathbb{P}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^{n} \zeta_i \geq \frac{1}{2}\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^{n} \zeta_i)) \geq \eta,\end{aligned}$$

where the second to last inequality holds since by (E.2), $\mathbb{E}_{\boldsymbol{\beta}=\mathbf{0},\lambda}(\sum_{i=1}^{n} \zeta_i) \geq 1$ for sufficiently large n.

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