The Annals of Applied Probability
2021, Vol. 31, No. 4, 1548–1597
https://doi.org/10.1214/20-AAP1626

ASYMPTOTIC DISTRIBUTION OF BERNOULLI QUADRATIC FORMS

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Consider the random quadratic form $T_n = \sum_{1 \le u < v \le n} a_{uv} X_u X_v$, where $((a_{uv}))_{1 \le u,v \le n}$ is a $\{0, 1\}$-valued symmetric matrix with zeros on the diagonal, and $X_1, X_2, \dots, X_n$ are i.i.d. $\mathrm{Ber}(p_n)$, with $p_n \in (0, 1)$. In this paper, we prove various characterization theorems about the limiting distribution of T_n, in the sparse regime, where $p_n \to 0$ such that $\mathbb{E}(T_n) = O(1)$. The main result is a decomposition theorem showing that distributional limits of T_n is the sum of three components: a mixture which consists of a quadratic function of independent Poisson variables; a linear Poisson mixture, where the mean of the mixture is itself a (possibly infinite) linear combination of independent Poisson random variables; and another independent Poisson component. This is accompanied with a universality result which allows us to replace the Bernoulli distribution with a large class of other discrete distributions. Another consequence of the general theorem is a necessary and sufficient condition for Poisson convergence, where an interesting second moment phenomenon emerges.

1. Introduction. Let $X_1, X_2, \dots, X_n$ be i.i.d. $\mathrm{Ber}(p_n)$, with $0 < p_n \ll 1$, where $a_n \ll b_n$ for any two positive sequences $\{a_n\}_{n \ge 1}$ and $\{b_n\}_{n \ge 1}$ of real numbers means $a_n = o(b_n)$. The well-known Poisson approximation to the Binomial distribution shows that, given a $\{0, 1\}$-valued sequence $a_1, a_2, \dots, a_n$, the *linear statistic*

$$L_n = \sum_{i=1}^{n} a_i X_i \xrightarrow{D} \mathrm{Pois}(\lambda),$$

whenever the mean $\mathbb{E} L_n = p_n \sum_{i=1}^n a_i \to \lambda$. Conversely, if $0 < p_n \ll 1$ is such that $p_n \sum_{i=1}^n a_i = O(1)$, then whenever L_n converges in distribution to a finite random variable, there exists $\lambda \ge 0$, such that L_n converges to $\mathrm{Pois}(\lambda)$. In other words, in the sparse regime, where $0 < p_n \ll 1$ is chosen such that $\mathbb{E}(L_n) = O(1)$, the Poisson distribution characterizes the limiting distribution of linear forms in Bernoulli variables.

In this paper we address the analogous question for quadratic forms in Bernoulli random variables: Given a $\{0, 1\}$-valued symmetric matrix $((a_{uv}))_{1 \le u,v \le n}$ with zeros on the diagonal, consider the *Bernoulli quadratic form*,

$$T_n = \sum_{1 \le u < v \le n} a_{uv} X_u X_v, \tag{1.1}$$

where, as before, $X_1, X_2, \dots, X_n$ are i.i.d. $\mathrm{Ber}(p_n)$. In this case, the *sparse regime* corresponds to choosing $0 < p_n \ll 1$, such that

$$\mathbb{E}(T_n) = p_n^2 \sum_{1 \le u < v \le n} a_{uv} = O(1). \tag{1.2}$$

Received January 2020; revised August 2020.
MSC2020 subject classifications. Primary 60C05, 60F05, 05D99; secondary 60H05, 05C80.
Key words and phrases. Combinatorial probability, limit theorems, moment phenomena, Poisson approximation, random quadratic forms.

In this regime the random variable $T_n = O_P(1)$, therefore, it has distributional limits along subsequences. In fact, using Stein's method for Poisson approximation [2–4, 11], it is easy to obtain various sufficient conditions on the matrix $((a_{uv}))_{1\le u,v\le n}$ for which T_n is asymptotically Poisson. However, unlike in the linear case, it is easy to construct matrices $((a_{uv}))_{1\le u,v\le n}$ for which T_n has a non-Poisson limit:

(1) Take $a_{uv} = 1$, for all $1 \le u \ne v \le n$, and choose $p_n = \lambda/n$ (for some $\lambda > 0$). Then $S_n = \sum_{u=1}^n X_u \xrightarrow{D} N \sim \text{Pois}(\lambda)$, and

$$T_n = \frac{1}{2} \sum_{1\le u\ne v\le n} X_u X_v = \binom{S_n}{2} \xrightarrow{D} \binom{N}{2}, \tag{1.3}$$

which is a quadratic function of a Poisson random variable.

(2) Take $b_n = \lfloor\sqrt{n}\rfloor$ and let $a_{uv} = a_{vu} = 1$, for $1 \le u \le b_n$ and $ub_n + 1 \le v \le ub_n + b_n$. Then

$$T_n = \sum_{u=1}^{b_n} X_u \sum_{v=ub_n+1}^{ub_n+b_n} X_v.$$

Here, choosing $p_n = \lambda/\sqrt{n}$ (for some $\lambda > 0$) ensures $\mathbb{E}(T_n) \to \lambda^2$. Then the random variables $J_u = \sum_{v=ub_n+1}^{ub_n+b_n} X_v \sim \text{Bin}(\lfloor\sqrt{n}\rfloor, \frac{\lambda}{\sqrt{n}})$, are independent for $1 \le u \le b_n$. This implies,

$$T_n = \sum_{u=1}^{b_n} X_u J_u \stackrel{D}{=} \text{Bin}\left(\lfloor\sqrt{n}\rfloor \sum_{u=1}^{b_n} X_u, \frac{\lambda}{\sqrt{n}}\right) \xrightarrow{D} \text{Pois}(\lambda N), \tag{1.4}$$

where $N \sim \text{Pois}(\lambda)$ (because $\sum_{u=1}^{b_n} X_u \xrightarrow{D} \text{Pois}(\lambda)$). In this case, the limit is a Poisson distribution with a random mean, that is, it is a *Poisson mixture* [23]. (Given a discrete random variable X, a Poisson mixture with mean X is denoted by $Z \sim \text{Pois}(X)$. More precisely, for $z \in \{0, 1, \dots, \}$, $\mathbb{P}(Z = z) = \mathbb{E}[\frac{e^{-X}X^z}{z!}]$.)

The different limits obtained in the examples above raise the question: *What are the possible limiting distributions of the Bernoulli quadratic form T_n in the sparse regime* (1.2)? In this paper, we prove a general decomposition theorem which allows us to express the limiting distribution of T_n as the sum of three components: a "quadratic component", which is a mixture driven by a bivariate Poisson stochastic integral; a "linear component" which is a Poisson mixture, where the mean of the mixture is itself a univariate Poisson stochastic integral; and an independent Poisson component (Theorem 1.1). Moreover, any distributional limit of T_n must belong to the closure of the class defined by the above decomposition (Theorem 1.2). This general result has several interesting consequences, such as a characterization theorem for dense matrices (Corollary 1.3), a second moment phenomenon for Poisson convergence (Corollary 1.4), and a universality phenomenon which allows us to replace the Bernoulli distribution with other discrete distributions (Corollary 1.5). In Section 2 we use these results to compute the limit of T_n in various natural examples.

1.1. *Asymptotic notation.* For positive sequences $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$, $a_n = O(b_n)$ means $a_n \le C_1 b_n$ and $a_n = \Theta(b_n)$ means $C_2 b_n \le a_n \le C_1 b_n$, for all n large enough and positive constants C_1, C_2. Similarly, for positive sequences $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$, $a_n \lesssim b_n$ means $a_n \le C_1 b_n$ and $a_n \gtrsim b_n$ means $a_n \ge C_2 b_n$, for all n large enough and positive constants C_1, C_2. Moreover, subscripts in the above notation, for example, $\lesssim_\circ$, denote that the hidden constants may depend on the subscripted parameters. Finally, $a_n \ll b_n$ and $a_n \gg b_n$ will mean $a_n = o(b_n)$ and $b_n = o(a_n)$, respectively.

1.2. *Limiting distribution of Bernoulli quadratic forms.* Hereafter, without loss of generality, we adopt the language of graph theory, and think of the matrix $((a_{uv}))_{1\le u,v\le n}$ as the adjacency matrix of an undirected simple graph on n vertices. To this end, let $\mathscr{G}_n$ denote the space of all simple undirected graphs on n vertices labeled by $[n] := \{1, 2, \dots, n\}$. Given a graph $G_n \in \mathscr{G}_n$ with adjacency matrix $A(G_n) = ((a_{uv}(G_n)))_{1\le u,v\le n}$, denote by $V(G_n)$ the set of vertices, and by $E(G_n)$ the set of edges of G_n, respectively. Then the Bernoulli quadratic form (1.1) (indexed by the graph G_n) becomes

$$T_n = \frac{1}{2}\sum_{1\le u,v\le n} a_{uv}(G_n)X_uX_v = \frac{1}{2}\boldsymbol{X}'A(G_n)\boldsymbol{X}, \tag{1.5}$$

where $X_1, X_2, \dots, X_n$ are i.i.d. $\mathrm{Ber}(p_n)$ and $\boldsymbol{X} = (X_1, X_2, \dots, X_n)'$. The sparse regime (1.2) translates to $0 < p_n \ll 1$ such that

$$\mathbb{E}[T_n] = |E(G_n)| p_n^2 = \Theta(1). \tag{1.6}$$

(Note that if $\mathbb{E}[T_n] = o(1)$, then $T_n \xrightarrow{P} 0$, hence, to obtain nondegenerate limiting distributions it suffices to consider the case $\mathbb{E}[T_n] = \Theta(1)$.)

REMARK 1.1. The statistic (1.5) arises naturally in several contexts, such as nonparametric two-sample tests [17], understanding coincidences [14], and motif frequency estimation in large networks [20]. For instance, in the study of coincidences T_n arises as a generalization of the birthday paradox [10, 12, 13], where the matrix $((a_{uv}))_{1\le u,v\le n}$ corresponds to the adjacency matrix of a friendship-network graph G_n, and one wishes to estimate the probability that there are two friends with birthday on a particular day (say January 31). Then taking $X_1, X_2, \dots, X_n$ i.i.d. Ber(1/365) (assuming birthdays are uniformly distributed over the year), T_n counts the number of pairs of friends with birthdays on January 31. This statistic also arises in the problem of estimating frequencies of motifs (small subgraphs) in large graphs [20, 25]. Here, given a large graph G_n, the goal is to efficiently estimate (without storing or searching over the entire graph) global characteristics, such as, the number of edges of G_n, by making local queries on G_n. In the subgraph sampling model [20, 29], where one has access to the random induced subgraph obtained by sampling each vertex of G_n independently with probability p_n, the statistic T_n/p_n^2, by (1.6), is an unbiased estimate of the number of edges in G_n.

Hereafter, we denote $r_n = 1/p_n$, and assume that the vertices of G_n are labelled in the nonincreasing order of the degrees $d_1 \ge d_2 \ge \cdots \ge d_n$, where d_v denotes the degree of the vertex labelled v. To describe the limiting distribution of T_n we need to consider limits of the sequence of matrices $((a_{uv}))_{1\le u,v\le n}$. This can be done using the framework of graph limit theory [8, 9, 24]. To this end, let $\mathcal{W}$ be the space of all symmetric measurable functions from $[0,\infty)^2 \to [0,1]$. Given a graph G_n (and a sequence $r_n \to \infty$), define the function $W_{G_n} \in \mathcal{W}$ as follows:

$$W_{G_n}(x,y) := \begin{cases} \mathbf{1}\{(\lceil xr_n\rceil, \lceil yr_n\rceil) \in E(G_n)\} & \text{for } x, y \in \left[0, \frac{n}{r_n}\right]^2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

Moreover, for a graph G_n, define the *normalized degree-function* as $d_{W_{G_n}}(x) = \int_0^\infty W_{G_n}(x,y)\,dy$. Note that

$$d_{W_{G_n}}(x) := \begin{cases} \dfrac{1}{r_n}\displaystyle\sum_{j=1}^n a_{\lceil xr_n\rceil j}(G_n) & \text{for } x \in \left[0, \frac{n}{r_n}\right], \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

DEFINITION 1.1 ([24]). For $K > 0$, the *cut-distance* between two functions $W_1, W_2 \in \mathcal{W}$, restricted to the domain $[0, K]^2$, is defined as,

$$\begin{aligned} &\|W_1 - W_2\|_{\square([0,K]^2)} \\ &:= \sup_{f,g:[0,K]\to[-1,1]} \left| \int_{[0,K]^2} \big(W_1(x,y) - W_2(x,y)\big) f(x) g(y)\, \mathrm{d}x\, \mathrm{d}y \right|. \end{aligned} \tag{1.9}$$

The *cut-metric* between two functions $W_1, W_2 \in \mathcal{W}$, restricted to the domain $[0, K]^2$, is defined as,

$$\delta_{\square([0,K]^2)}(W_1, W_2) := \inf_{\psi} \|W_1^{\psi} - W_2\|_{\square([0,K]^2)}, \tag{1.10}$$

with the infimum taken over all measure-preserving bijections $\psi : [0, K] \to [0, K]$, and $W_1^{\psi}(x, y) := W_1(\psi(x), \psi(y))$, for $x, y \in [0, K]$.

Equipped with the definitions above we can now state our main theorem. To this end, for $p \geq 1$ and a Borel set $\mathcal{K} \subseteq \mathbb{R}^d$ denote by $L_p(\mathcal{K})$ the set of all measurable functions from $\mathcal{K} \to \mathbb{R}$ such that $\int_{\mathcal{K}} |f(\boldsymbol{x})|^p\, \mathrm{d}\boldsymbol{x} < \infty$. Also, given a function $f \in L_1([0, \infty)^d)$, $\int f(x_1, x_2, \ldots, x_d) \prod_{a=1}^{d} \mathrm{d}N(x_a)$, will denote the multiple Itô stochastic integral of f with respect to the homogeneous Poisson process of rate 1, $\{N(t), t \geq 0\}$. The precise definition of stochastic integration with respect to a Poisson process and methods for computing them are given in Appendix B.

THEOREM 1.1. *Let $X_1, X_2, \ldots, X_n$ be i.i.d.* $\mathrm{Ber}(p_n)$ *and suppose $\{G_n\}_{n\geq 1}$ is a sequence of graphs such that (1.6) is satisfied. Assume that the vertices of G_n are labelled $\{1, 2, \ldots, n\}$ in nonincreasing order of the degrees and the following hold*:

(a) $\lim_{K\to\infty} \lim_{n\to\infty} \frac{1}{2} \int_K^{\infty} \int_K^{\infty} W_{G_n}(x, y)\, \mathrm{d}x\, \mathrm{d}y = \lambda_0$.

(b) *There exists a function $W \in \mathcal{W}$, such that, for $K > 0$ large enough*,

$$\lim_{n\to\infty} \|W_{G_n} - W\|_{\square([0,K]^2)} = 0. \tag{1.11}$$

(c) *There exists a function $d : [0, \infty) \to [0, \infty)$ in $L_1([0, \infty))$, such that, for $K, M > 0$ large enough*,

$$\lim_{n\to\infty} \int_0^K \big| d_{W_{G_n}}(x) \mathbf{1}\{d_{W_{G_n}}(x) \leq M\} - d(x) \mathbf{1}\{d(x) \leq M\} \big|\, \mathrm{d}x = 0. \tag{1.12}$$

Then

$$T_n := \frac{1}{2} \sum_{1\leq u,v\leq n} a_{uv}(G_n) X_u X_v \xrightarrow{D} Q_1 + Q_2 + Q_3, \tag{1.13}$$

where

– $Q_3 \sim \mathrm{Pois}(\lambda_0)$ *and Q_3 is independent of (Q_1, Q_2).*
– *The joint moment generating function of (Q_1, Q_2) is given by*: *For $t_1, t_2 \geq 0$,*

$$\begin{aligned} &\mathbb{E} \exp\{-t_1 Q_1 - t_2 Q_2\} \\ &= \mathbb{E} \exp\left\{ \frac{1}{2} \int_0^{\infty} \int_0^{\infty} \phi_{W,t_1}(x, y)\, \mathrm{d}N(x)\, \mathrm{d}N(y) - \hat{t}_2 \int_0^{\infty} \Delta(x)\, \mathrm{d}N(x) \right\}, \end{aligned} \tag{1.14}$$

with

– $\hat{t}_2 := (1 - e^{-t_2})$,
– $\int_{[0,\infty)^2} W(x, y)\, \mathrm{d}x\, \mathrm{d}y < \infty$,

– $\Delta(x) := d(x) - \int_0^\infty W(x,y)\,\mathrm{d}y$,
– $\{N(t), t \geq 0\}$ *is a homogenous Poisson process of rate* 1, *and*
– $\phi_{W,t_1}(x,y) := \log(1 - W(x,y) + W(x,y)e^{-t_1})$.

The proof of this result is given in Section 3. The proof proceeds by decomposing the graph into three components based on the degree of the vertices (as explained below), and then approximating each of the components in moments by an appropriately constructed random variable with more independence structure, for which the joint asymptotic distribution can be explicitly computed. (A detailed overview of the proof of Theorem 1.1 is given in the beginning of Section 3.) The three components give rise to the following three terms in the limiting distribution of T_n:

- A *quadratic component* Q_1 whose moment generating function is given in terms of a bivariate stochastic integral. This is the contribution to T_n from the "dense core" of the graph, that is, edges between the "high-degree" vertices (degree greater than $\frac{r_n}{K}$) of G_n.
- A *linear component* Q_2, which is the contribution to T_n from the edges between the "high-degree" and "low-degree" vertices (degree less than $\frac{r_n}{K}$) of G_n. Note that the marginal moment generating function of Q_2 is

$$\mathbb{E}\exp\{-t_2 Q_2\} = \mathbb{E}\exp\left\{-(1-e^{-t_2})\int_0^\infty \Delta(x)\,\mathrm{d}N(x)\right\}. \tag{1.15}$$

  By comparing moment generating functions, it is easy to see that $Q_2 \sim \mathrm{Pois}(R_2)$, where $R_2 = \int_0^\infty \Delta(x)\,\mathrm{d}N(x)$ is a univariate Poisson stochastic integral. This shows that marginally Q_2 is a Poisson mixture, where the mixing distribution is a (possibly) infinite linear combination of independent Poisson random variables.
- An *independent Poisson component* Q_3, which is the contribution from the edges between the "low-degree" vertices of G_n.

REMARK 1.2. Even though (1.14) often characterizes the limit of T_n (as shown in Theorem 1.2, Corollary 1.3, and Corollary 1.4 below), the conditions in Theorem 1.1 can be slightly relaxed in a few ways:

(1) It will be evident from the proof of Theorem 1.1 that it suffices to assume (1.11) holds, not for all K large enough, but along any diverging sequence $K_s \to \infty$. Similarly, condition (1.12) only needs to hold along diverging sequences of K and M. In fact, we show later in Observation 3.2 that an easy sufficient condition for (1.12) to hold along a certain diverging sequence of M is

$$\lim_{n\to\infty} \|d_{W_{G_n}} - d\|_{L_1([0,K])} = 0.$$

We will often use the condition above to verify (1.12). However, the truncated condition in (1.12) is, in general, necessary to include graphs with a few high-degree vertices.

(2) Another relaxation, which will again be clear from the proof of Theorem 1.1, is to assume (1.11) and (1.12) hold along a common bijection (permutation of the vertices) from $[0,K] \to [0,K]$ (see Lemma 3.5 for a precise statement). Marginally, this allows one to replace the cut-distance $\|\cdot\|_{\square([0,K]^2)}$ in (1.11) with the cut-metric $\delta_{\square([0,K]^2)}$. This generalization will be important for establishing the necessity of the conditions and characterizing the limits of T_n (in Theorem 1.2 and Corollary 1.3 below). Nevertheless, to avoid notational clutter, we present Theorem 1.1 under the slightly weaker condition, and discuss this generalization as part of the proof in Section 3.4.

Given the above discussion, it is natural to wonder whether the conditions (1.11) and (1.12) are necessary for the convergence of T_n. More generally, one can ask what are the possible limiting distributions of T_n? It is easy to construct examples where T_n does not converge in distribution, when the conditions of Theorem 1.1 are not satisfied (see Example 7). However, the question of determining all possible limiting distributions of T_n is more delicate. In the theorem below, we answer this question by showing that whenever T_n has a distributional limit, it must belong to the closure of limits of the form (1.14). To make this precise, denote by $\mathcal{F}$ the collection of all functions $d : [0, \infty) \to [0, \infty)$ in $L_1([0, \infty))$, and consider the following definition:

DEFINITION 1.2. For $\mathcal{W}$ and $\mathcal{F}$ as above, define $\mathcal{P}(\mathcal{W}, \mathcal{F})$ to be the collection of all probability measures μ on $\mathbb{Z}_+ \cup \{0\}$, such that if $J \sim \mu$, then

$$J \stackrel{D}{=} J_1 + J_2,$$

where the joint moment generating function of (J_1, J_2) is given by the RHS of (1.14), for some function $W \in \mathcal{W}$ with $\int_0^\infty \int_0^\infty W(x, y)\,\mathrm{d}x\,\mathrm{d}y < \infty$ and some function $d \in \mathcal{F}$, such that $\Delta(x) = d(x) - \int_0^\infty W(x, y)\,\mathrm{d}y \geq 0$, for all $x \in [0, \infty)$. Finally, denote by $\overline{\mathcal{P}}(\mathcal{W}, \mathcal{F})$ the closure of $\mathcal{P}(\mathcal{W}, \mathcal{F})$ under weak convergence. (More precisely, a probability measure μ on $\mathbb{Z}_+ \cup \{0\}$ belongs to $\overline{\mathcal{P}}(\mathcal{W}, \mathcal{F})$ if and only if there exists a sequence of probability measures $\{\mu_s\}_{s\geq 1}$, with $\mu_s \in \mathcal{P}(\mathcal{W}, \mathcal{F})$, such that μ_s converges weakly (in distribution) to μ, as $s \to \infty$.)

The following theorem shows that whenever T_n has a distributional limit, it has a component which belongs to $\overline{\mathcal{P}}(\mathcal{W}, \mathcal{F})$ plus an independent Poisson random variable.

THEOREM 1.2. *Suppose* (1.6) *holds and the random variable T_n converges in distribution to a random variable T. Then $T \stackrel{D}{=} J + J_0$, where $J \in \overline{\mathcal{P}}(\mathcal{W}, \mathcal{F})$, $J_0 \sim$* Pois(λ), *for some $\lambda \geq 0$, and J_0 is independent of J.*

The proof of the above theorem is given in Section 4. We compute the limit of T_n in different examples in Section 2. Interestingly, in all the examples constructed in Section 4 the limiting distribution of T_n belongs to the class $\mathcal{P}(\mathcal{W}, \mathcal{F})$ itself. This leaves open the intriguing question of whether there are distributional limits of T_n which are in $\overline{\mathcal{P}}(\mathcal{W}, \mathcal{F})$ but not in $\mathcal{P}(\mathcal{W}, \mathcal{F})$.

1.3. *Consequences of Theorem* 1.1. The limiting distribution in Theorem 1.1 simplifies if the graph sequence $\{G_n\}_{n\geq 1}$ has some special structures.

We begin with the case when the graph is dense. Recall a sequence of graphs $\{G_n\}_{n\geq 1}$ is said to be *dense*, if $|E(G_n)| \geq Cn^2$, for some constant $C > 0$, when n is large enough. In this case, the assumption (1.11) characterizes all limits of T_n. Here, the linear mixture and the Poisson components vanish, and the limit of T_n is determined by the quadratic component.

COROLLARY 1.3 (Dense Graphs). *Let $X_1, X_2, \ldots, X_n$ be i.i.d.* Ber(p_n) *and suppose $\{G_n\}_{n\geq 1}$ is a sequence of dense graphs such that* (1.6) *holds.*

(a) *Suppose there exists a function $W \in \mathcal{W}$, such that, for $K > 0$ large enough, $\lim_{n\to\infty} \|W_{G_n} - W\|_{\square([0,K]^2)} = 0$. Then W vanishes outside a compact rectangle $[0, a]^2$ for some finite $a \geq 0$, and $T_n \xrightarrow{D} Q_1$, where*

$$\mathbb{E}\exp\{-t_1 Q_1\} = \mathbb{E}\exp\left\{\frac{1}{2}\int_0^a \int_0^a \phi_{W,t_1}(x, y)\,\mathrm{d}N(x)\,\mathrm{d}N(y)\right\}, \tag{1.16}$$

with $t_1 \geq 0$, $\phi_{W,t_1}(x, y) := \log(1 - W(x, y) + W(x, y)e^{-t_1})$, *and* $\{N(t), t \geq 0\}$ *is a homogenous Poisson process of rate* 1.

(b) *Conversely, suppose* $\{G_n\}_{n\geq 1}$ *is a sequence of dense graphs such that* (1.6) *holds, and* T_n *converges in distribution. Then the limit is necessarily of the form* (1.16), *for some function* $W \in \mathcal{W}$ *which vanishes outside* $[0, a]^2$ *for some finite* $a \geq 0$.

The proof of Corollary 1.3 is given in Section 5. In Section 2, we compute the limit in (1.16) in various examples.

Another consequence of Theorem 1.1, is a characterization of when the limiting distribution of T_n is a Poisson random variable. This reveals an interesting truncated *second moment phenomenon*, that is, the convergence of the first two moments of a truncated version of T_n determines the convergence in distribution to a Poisson distribution. To this end, for any $M > 0$, define $X_{u,M} := X_u \mathbf{1}\{d_u \leq M r_n\}$ and

$$T_{n,M} = \sum_{(u,v)\in E(G_n)} X_{u,M} X_{v,M}. \tag{1.17}$$

Moreover, for a doubly indexed sequence of real numbers $\{a_{n,m}\}_{n,m\geq 1}$, the double limit $\lim_{m\to\infty} \lim_{n\to\infty} a_{n,m} = a$, means

$$\limsup_{m\to\infty} \limsup_{n\to\infty} a_{n,m} = \liminf_{m\to\infty} \liminf_{n\to\infty} a_{n,m} = a.$$

COROLLARY 1.4 (Truncated second moment phenomenon for Poisson approximation). *Let* $X_1, X_2, \dots, X_n$ *be i.i.d.* $\mathrm{Ber}(p_n)$ *and suppose* $\{G_n\}_{n\geq 1}$ *is a sequence of graphs such that* (1.6) *holds. Then the following are equivalent.*

(a) $T_n \xrightarrow{D} \mathrm{Pois}(\lambda)$.
(b) $\lim_{M\to\infty} \lim_{n\to\infty} \mathbb{E} T_{n,M} = \lambda$ *and* $\lim_{M\to\infty} \lim_{n\to\infty} \mathrm{Var}(T_{n,M}) = \lambda$.
(c) *The assumptions of Theorem* 1.1 *hold with* $W = 0$, $d = 0$, *and* $\lambda_0 = \lambda$.

The corollary above shows that the Poisson convergence of T_n is characterized by the convergence of just the first two truncated moments of T_n (the proof is given in Section 5.2). In fact, a simpler sufficient condition for the Poisson convergence of T_n is the convergence of the first two (un-truncated) moments of T_n, that is, $T_n \xrightarrow{D} \mathrm{Pois}(\lambda)$ whenever $\lim_{n\to\infty} \mathbb{E} T_n = \lambda$ and $\lim_{n\to\infty} \mathrm{Var}(T_n) = \lambda$, which can also be directly proved using the well-known Stein's method for Poisson approximation based on dependency graphs [3, 4, 11]. However, convergence of the first two un-truncated moments is clearly not necessary for the Poisson convergence of T_n, as shown in Example 3. To obtain the necessary and sufficient condition for the Poisson convergence of T_n, we need to consider the truncated conditions as in Corollary 1.4 above.

REMARK 1.3. This second moment phenomenon for the Poisson distribution for random quadratic forms complements the well-known fourth-moment phenomenon, which asserts that the limiting normal distribution of certain centered homogeneous forms is implied by the convergence of the corresponding sequence of fourth moments (refer to Nourdin et al. [26, 27] and the references therein, for general fourth-moment theorems and invariance principles and [6, 15] for an example of this phenomenon in random graph coloring). As in the fourth-moment phenomenon for normal approximation, this second moment phenomenon for Poisson approximation exhibits universality (see Section 1.4 below), and we expect this phenomenon to extend beyond the quadratic to general integer-valued homogeneous sums.

1.4. *Universality.* It is natural to ask what happens if one considers quadratic forms in other integer-valued random variables (not necessarily Bernoulli). More precisely, if $X_1, X_2, \dots, X_n$ are i.i.d. nonnegative integer valued random variables with distribution function F_n, then (similar to (1.5)) the F_n-quadratic form, indexed by a graph G_n, is defined as

$$T_n = \frac{1}{2}\sum_{1\le u,v\le n} a_{uv}(G_n)X_uX_v = \frac{1}{2}\boldsymbol{X}'A(G_n)\boldsymbol{X}, \tag{1.18}$$

where $\boldsymbol{X} = (X_1, X_2, \dots, X_n)'$. It turns out that the limiting distribution of a general F_n-quadratic form exhibits a universality, whenever X_1 has the property $\frac{\mathbb{E}X_1}{\mathbb{P}(X_1=1)} = 1 + o(1)$, that is, the contribution to the expectation is essentially determined by $\mathbb{P}(X_1 = 1)$.

COROLLARY 1.5. *Suppose $\{X_v\}_{1\le v\le n}$ are i.i.d. nonnegative integer valued random variables with $p_n := \mathbb{P}(X_1 = 1) \to 0$, such that $|E(G_n)|p_n^2 = \Theta(1)$ (as in (1.6)) and $\lim_{n\to\infty}\frac{1}{p_n}\mathbb{E}X_1 = 1$. Then if the graph sequence $\{G_n\}_{n\ge1}$ satisfies the assumptions of Theorem 1.1,*

$$T_n \xrightarrow{D} Q_1 + Q_2 + Q_3,$$

where T_n is as defined in (1.18) and Q_1, Q_2, and Q_3 are as in Theorem 1.1.

This result shows that Theorem 1.1, and, as a consequence, Corollary 1.3 and Corollary 1.4, extend beyond the (sparse) Bernoulli, to include cases like the sparse Poisson, binomial, negative binomial, and hypergeometric, among others, and complements the well-known universality of the Weiner chaos for centered homogeneous sums [27].

1. *Sparse Poisson*: Suppose $X_1, X_2, \dots, X_n$ are i.i.d. Pois(θ_n), where $\theta_n \to 0$. In this case, $\mathbb{P}(X_1 = 1) = \theta_n e^{-\theta_n} \to 0$ and $\mathbb{E}X_1 = \theta_n$, and so $\frac{\mathbb{E}X_1}{\mathbb{P}(X_1=1)} = e^{\theta_n} \to 1$, as required in Corollary 1.5.

2. *Sparse binomial*: Suppose $X_1, X_2, \dots, X_n$ are i.i.d. Ber(m_n, θ_n), where m_n and θ_n satisfy $m_n\theta_n \to 0$. In this case, $\mathbb{P}(X_1 = 1) = m_n\theta_n(1-\theta_n)^{m_n-1} \to 0$, and $\mathbb{E}X_1 = m_n\theta_n$, and so $\frac{\mathbb{E}X_1}{\mathbb{P}(X_1=1)} = \frac{1}{(1-\theta_n)^{m_n-1}} \to 1$, as required in Corollary 1.5.

3. *Sparse negative binomial*: Suppose $X_1, X_2, \dots, X_n$ are i.i.d. NB(m_n, θ_n) with

$$\mathbb{P}(X_1 = r) = \binom{r + m_n - 1}{r}(1-\theta_n)^{m_n}\theta_n^r \quad \text{for } r = 0, 1, \dots,$$

where m_n and θ_n satisfy $m_n\theta_n \to 0$. In this case, $\mathbb{P}(X_1 = 1) = m_n\theta_n(1-\theta_n)^{m_n} \to 0$, and $\mathbb{E}X_1 = \frac{m_n\theta_n}{1-\theta_n}$, and so $\frac{\mathbb{E}X_1}{\mathbb{P}(X_1=1)} = \frac{1}{(1-\theta_n)^{m_n+1}} \to 1$, as required in Corollary 1.5.

4. *Sparse hypergeometric*: Suppose $X_1, X_2, \dots, X_n$ are i.i.d. HGeom(N_n, K_n, m_n) with

$$\mathbb{P}(X_1 = r) = \frac{\binom{K_n}{r}\binom{N_n-K_n}{m_n-r}}{\binom{N_n}{m_n}} \quad \text{for } r \in \{\max(0, m_n + K_n - N_n), \dots, \min(m_n, K_n)\},$$

where (N_n, K_n, m_n) satisfy $N_n \to \infty$, $\frac{m_nK_n}{N_n} \to 0$, and $\min(m_n, K_n) \ge 1$. This implies $N_n - (m_n + K_n) \to \infty$, and so, for all n large, 0 and 1 are both in the support of X_1. Further,

$$\mathbb{P}(X_1 = 1) = \frac{K_n\binom{N_n-K_n}{m_n-1}}{\binom{N_n}{m_n}} = \frac{m_nK_n}{N_n}\cdot\frac{(N_n-K_n)!(N_n-m_n)!}{(N_n-K_n-m_n+1)!(N_n-1)!}$$

$$= \frac{m_nK_n}{N_n}\prod_{s=1}^{m_n-1}\frac{N_n-K_n+1-s}{N_n-s} = \frac{m_nK_n}{N_n}\cdot a_n,$$

where

$$1 \geq a_n = \prod_{s=1}^{m_n-1}\left(1-\frac{K_n-1}{N_n-s}\right) \geq \left(1-\frac{K_n-1}{N_n-m_n+1}\right)^{m_n-1} \to 1,$$

since $\frac{m_n K_n}{N_n} \to 0$. Thus, $\mathbb{P}(X_n = 1) \to 0$ and $\frac{\mathbb{E}X_1}{\mathbb{P}(X_1=1)} = \frac{1}{a_n} \to 1$, as required in Corollary 1.5.

1.5. *Organization.* The rest of the paper is organized as follows: In Section 2, we compute the limiting distribution in various examples. The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 3 and Section 4, respectively. The proofs of Corollaries 1.3, 1.4, and 1.5 are given in Section 5. Details about Poisson stochastic integrals and other technical lemmas are discussed in the Appendix.

2. Examples. In this section we use Theorem 1.1 to compute the limiting distribution of T_n for various graph sequences. In the examples below, we will often construct graph sequences $G_n = (V(G_n), E(G_n))$, where $|V(G_n)| \neq n$, but $|V(G_n)| \to \infty$, as $n \to \infty$. In such cases, the definitions in (1.7) and (1.8) have to be modified, with the number of vertices n replaced by $|V(G_n)|$ appropriately, following which the results hold verbatim.

We begin with an application of Corollary 1.3 for dense block graphons.

EXAMPLE 1 (Dense block graphons). Let $X_1, X_2, \ldots, X_n$ be i.i.d. Ber(λ/n), for some $\lambda > 0$. Fix $\kappa > 0$ and consider a sequence of dense graphs G_n converging in cut-metric to the B-block function $f : [0,\kappa]^2 \to [0,1]$, given by

$$f(x,y) = \begin{cases} b_{jj} & \text{if } x, y \in [c_{j-1}, c_j], \text{ for some } j \in [B], \\ b_{jj'} & \text{if } x \in [c_{j-1}, c_j], y \in [c_{j'-1}, c_{j'}], \text{ for some } j \neq j' \in [B], \end{cases} \tag{2.1}$$

where $c_0 = 0$, $c_B = \kappa$, $[B] := \{1, 2, \ldots, B\}$, and the constants $\{b_{jj'}, j, j' \in [B]\}$, and $c_1, c_2, \ldots, c_B$ are chosen such that $b_{jj'} = b_{j'j}$, for $j \neq j' \in [B]$ and $\int_0^\kappa \int_0^\kappa f(x,y)\,\mathrm{d}x\,\mathrm{d}y > 0$. (This is obtained as the graph limit of a stochastic block model (SBM) on $\lceil n\kappa \rceil$ vertices and B blocks, where the edge (u,v) exists independently with probability $b_{jj'}$, when $u \in [\lceil nc_{j-1}\rceil, \lceil nc_j \rceil]$ and $v \in [\lceil nc_{j'-1}\rceil, \lceil nc_{j'}\rceil]$.) Now, given $t_1 \geq 0$, recall $\phi_{f,t_1}(x,y) := \log(1 - f(x,y) + f(x,y)e^{-t_1})$. Then by Example 8 and (1.16), for $t_1 \geq 0$,

$$\mathbb{E}\exp\{-t_1 Q_1\} = \mathbb{E}\exp\left\{\sum_{j=1}^{B} \psi_{f,t_1}(j,j)\binom{N_j}{2} + \sum_{1 \leq j < j' \leq B} \psi_{f,t_1}(j,j')N_j N_{j'}\right\}, \tag{2.2}$$

where $\psi_{f,t_1}(j,j') := \log(1 - b_{jj'} + b_{jj'}e^{-t_1})$, for $j, j' \in [B]$, and $\{N_1, N_2, \ldots, N_B\}$ are independent with $N_j \sim \mathrm{Pois}(c_j - c_{j-1})$. Now, consider the random variable,

$$Q_1' := \sum_{j=1}^{B} \eta_{jj} + \sum_{1 \leq j < j' \leq B} \eta_{jj'}, \tag{2.3}$$

where $\eta_{jj} \sim \mathrm{Bin}(\binom{N_j}{2}, b_{jj})$, $\eta_{jj'} \sim \mathrm{Bin}(N_j N_{j'}, b_{jj'})$ for $j \neq j'$, and the collection $\{\eta_{jj'} : 1 \leq j, j' \leq B\}$ are independent given $\{N_1, N_2, \ldots, N_B\}$. (Given $q \in [0,1]$ and a discrete random variable X, $Z \sim \mathrm{Bin}(X, q)$ denotes a Binomial distribution with a random number of trials X. More precisely, for $z \in \{0, 1, \ldots, \}$, $\mathbb{P}(Z = z) = \mathbb{E}[\binom{X}{z} q^z (1-q)^{X-z}]$.) Then it follows that, for $t_1 \geq 0$,

$$\begin{aligned}
&\mathbb{E}\exp\{-t_1 Q_1' | \{N_1, N_2, \ldots, N_B\}\} \\
&\quad = \prod_{j=1}^{B} (1 - b_{jj} + b_{jj} e^{-t_1})^{\binom{N_j}{2}} \prod_{1 \leq j < j' \leq B} (1 - b_{jj'} + b_{jj'} e^{-t_1})^{N_j N_{j'}} \\
&\quad = \exp\left\{ \sum_{j=1}^{B} \psi_{f,t_1}(j,j) \binom{N_j}{2} + \sum_{1 \leq j < j' \leq B} \psi_{f,t_1}(j,j') N_j N_{j'} \right\}.
\end{aligned} \tag{2.4}$$

This implies, for all $t_1 \geq 0$, $\mathbb{E}\exp\{-t_1 Q_1'\} = \mathbb{E}\exp\{-t_1 Q_1\}$, that is, $Q_1 \stackrel{D}{=} Q_1'$, which shows, if $\{G_n\}_{n\geq 1}$ is a sequence of graphs converging to the B-block function f (as in (2.1)), then $T_n \stackrel{D}{\to} Q_1'$, as defined in (2.3). For specific choices of f this further simplifies. For example, suppose $\{G_n\}_{n\geq 1}$ is a sequence of graphs converging to the 2-block function

$$W(x,y) = \begin{cases} b_{11} & \text{for } x, y \in [0, \alpha], \\ b_{22} & \text{for } x, y \in [\alpha, 1], \\ b_{12} & \text{otherwise.} \end{cases}$$

Then,

$$T_n \stackrel{D}{\to} \operatorname{Bin}\left(\binom{N_1}{2}, b_{11}\right) + \operatorname{Bin}(N_1 N_2, b_{12}) + \operatorname{Bin}\left(\binom{N_2}{2}, b_{22}\right), \tag{2.5}$$

where $N_1 \sim \operatorname{Pois}(\alpha\lambda)$, $N_2 \sim \operatorname{Pois}((1-\alpha)\lambda)$ are independent, and the three summands in (2.5) are independent given N_1, N_2. This includes as special cases, the Erdős–Rényi graph and the random bipartite graph. (By a simple conditioning argument, Corollaries 1.3 and 1.4 can be extended to random graphs by conditioning on the graph, under the assumption that the graph and its coloring are jointly independent (see, e.g., [7], Lemma 4.1). In particular, the convergence of T_n in Corollary 1.3 and Corollary 1.4 hold whenever the required conditions hold in probability.)

- *Dense Erdős–Rényi graphs*: When $\alpha = 1$, the graphon W reduces to the constant function b_{11}. This is attained as the graphon limit when $G_n \sim G(n, b_{11})$ is a sequence of Erdős–Rényi random graphs such that $b_{11} \in (0, 1]$ is fixed. In this case, (2.5) simplifies to

$$T_n \stackrel{D}{\to} \operatorname{Bin}\left(\binom{N_1}{2}, b_{11}\right), \tag{2.6}$$

  where $N_1 \sim \operatorname{Pois}(\lambda)$. In particular, if $b_{11} = 1$, that is, $G_n = K_n$ is the complete graph, then $T_n \stackrel{D}{\to} \binom{N_1}{2}$ (recall (1.3)).
- *Random bipartite graphs*: When $b_{11} = b_{22} = 0$, then this is attained as the limit of the random bipartite graph $G_n \sim G(\lceil \alpha n \rceil, \lceil (1-\alpha)n \rceil, b_{12})$, with edge probability $b_{12} \in (0, 1]$. Then, (2.5) simplifies to

$$T_n \stackrel{D}{\to} \operatorname{Bin}(N_1 N_2, b_{12}),$$

  where $N_1 \sim \operatorname{Pois}(\alpha\lambda)$, $N_2 \sim \operatorname{Pois}((1-\alpha)\lambda)$ are independent.

For more sparser graphs, the limiting distribution is often a Poisson, and Corollary 1.4 can be applied.

EXAMPLE 2 (Nondense approximately regular graphs). Let $X_1, X_2, \ldots, X_n$ be i.i.d. $\mathrm{Ber}(p_n)$ and $\{G_n\}_{n\geq 1}$ be a sequence of graphs such that

$$\lim_{n\to\infty} |E(G_n)| p_n^2 = \lambda \quad \text{and} \quad \Delta(G_n) := \max_{v \in V(G_n)} d_v = o(r_n). \tag{2.7}$$

Then for any $\varepsilon > 0$ there exists n large enough, such that $d_v \leq \varepsilon r_n$, for all $v \in V(G_n)$. Hence, for any $M \geq 1$ and n large enough $T_n = T_{n,M}$. This implies,

$$\lim_{M\to\infty} \lim_{n\to\infty} \mathbb{E}T_{n,M} = \lim_{n\to\infty} \mathbb{E}T_n = \lim_{n\to\infty} |E(G_n)| p_n^2 \to \lambda. \tag{2.8}$$

Moreover, for all large n,

$$\begin{aligned} \mathrm{Var}(T_{n,M}) &= \mathrm{Var}(T_n) \\ &= |E(G_n)| \mathrm{Var}(X_1X_2) + 2N(K_{1,2}, G_n) \mathrm{Cov}(X_1X_2, X_1X_3), \end{aligned} \tag{2.9}$$

where $N(K_{1,2}, G_n) = \sum_{v=1}^n \binom{d_v}{2}$ denotes the number of 2-stars in the graph G_n. Note that $\mathrm{Var}(X_1X_2) = p_n^2 - p_n^4$ and $\mathrm{Cov}(X_1X_2, X_1X_3) = p_n^3 - p_n^4$. Therefore,

$$\lim_{n\to\infty} |E(G_n)| \mathrm{Var}(X_1X_2) = \lambda,$$

and using $N(K_{1,2}, G_n) \leq \varepsilon |E(G_n)| r_n$, gives $\limsup_{n\to\infty} N(K_{1,2}, G_n) \mathrm{Cov}(X_1X_2, X_1X_3) \leq \varepsilon$. Then (2.9) implies,

$$\lim_{M\to\infty} \lim_{n\to\infty} \mathrm{Var}(T_{n,M}) = \lambda,$$

since ε is arbitrary. This combined with (2.8) and Corollary 1.4 shows that $T_n \xrightarrow{D} \mathrm{Pois}(\lambda)$, whenever (2.7) holds. This derives the limiting distribution of nondense (that is, $|E(G_n)| = o(n^2)$), "approximately" regular graphs.

- *Nondense regular graphs*: Let G_n be a sequence of d-regular graphs such that $d = o(n)$ and $\frac{nd}{2} p_n^2 \to \lambda$. Then $r_n = 1/p_n = \Theta(\sqrt{nd})$ and the maximum degree $d = o(r_n)$. Therefore, by the argument above, $T_n \xrightarrow{D} \mathrm{Pois}(\lambda)$.
- *Nondense Erdős–Rényi graphs*: Let $G_n \sim G(n, q_n)$ be a sequence of Erdős–Rényi random graphs such that $\frac{\log n}{n} \ll q_n \ll 1$ and $\frac{n^2 q_n}{2} p_n^2 \to \lambda$. Then $r_n = 1/p_n = \Theta(n\sqrt{q_n})$ and the maximum degree $\Delta(G_n) = (1 + o_P(1))nq_n = o(r_n)$ [21]. Therefore, by the argument above, $T_n \xrightarrow{D} \mathrm{Pois}(\lambda)$.

In the example above, the maximum degree of G_n is "small", and, as a result, condition (b) in Corollary 1.4 holds for the original (un-truncated) random variable T_n, as well (see (2.8) and (2.9)). However, the truncation is necessary when there are few vertices with "large" degree, as illustrated below.

EXAMPLE 3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $\mathrm{Ber}(\gamma/\sqrt{n})$. We consider two examples where truncation matters:

(1) Let $G_n = K_{1,n}$ be the n-star. Then $|E(G_n)| = n$ and (1.6) is satisfied. In this case, since the degree of the central vertex of the star is $n \gg M\sqrt{n}$, for any $M \geq 1$, $T_{n,M}$ is identically zero. Hence, condition (b) in Corollary 1.4 holds with $\lambda = 0$, which implies $T_n \xrightarrow{P} 0$.

(2) To get a nonzero limiting distribution, take G_n to be the disjoint union of a n-star $K_{1,n}$ and n disjoint edges $(\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\})$. As before, there is no contribution to $T_{n,M}$ from the star-graph, and

$$T_{n,M} = \sum_{j=1}^n X_{a_j} X_{b_j}.$$

This is the sum of independent indicators $Z_j = X_{a_j} X_{b_j} \sim \mathrm{Ber}(\gamma^2/n)$, and hence $\mathbb{E}T_{n,M} = \gamma^2$ and $\mathrm{Var}(T_{n,M}) \to \gamma^2$. Then, by Corollary 1.4, $T_n \xrightarrow{D} \mathrm{Pois}(\gamma^2)$.

Note that, as expected, in both the examples above the convergence is not in L_1: in (1) $\mathbb{E}T_n = \gamma^2$ and in (2) $\mathbb{E}T_n = 2\gamma^2$.

The Poisson mixture arises in the limit of T_n for bipartite graph which have many "high" degree vertices on one of the sides, and is best illustrated by considering a disjoint union of star graphs.

EXAMPLE 4 (Disjoint union of stars). Let G_n be the disjoint union of n isomorphic copies of the n-star $K_{1,n}^{(1)}, \dots, K_{1,n}^{(n)}$. Note that, $|V(G_n)| = n^2 + n$ and $|E(G_n)| = n^2$. Label the central vertices of the stars $1, 2, \dots, n$, the leaves of the vertex 1 as $n+1, \dots, 2n$, the leaves of the vertex 2 as $2n+1, \dots, 3n$, and so on. Let $X_1, X_2, \dots, X_n$ be i.i.d. $\mathrm{Ber}(1/n)$, which ensures $\mathbb{E}(T_n) = \frac{|E(G_n)|}{n^2} = 1$. Fix $K \geq 1$, denote by $G_{n,K}$ the induced subgraph of G_n on the first $\lceil Kn \rceil$ vertices. Then

$$\int_0^K \int_0^K W_{G_n}(x, y) \,\mathrm{d}x \,\mathrm{d}y \leq \frac{2|E(G_{n,K})|}{n^2} \lesssim \frac{K}{n} \to 0,$$

as $n \to \infty$. Therefore, $\|W_{G_n}\|_{\square([0,K]^2)} \lesssim \|W_{G_n}\|_{L_1([0,K]^2)} \to 0$, that is, condition (1.11) holds with $W = 0$. Moreover, for every $K \geq 1$, there are no edges in G_n between the vertices $\{\lceil Kn \rceil + 1, \dots, n^2\}$, which means $\lim_{K\to\infty} \lim_{n\to\infty} \int_K^\infty \int_K^\infty W_{G_n}(x, y) \,\mathrm{d}x \,\mathrm{d}y = 0$. Finally, the normalized degree-functional is (recall (1.8)),

$$d_{W_{G_n}}(x) := \frac{1}{n} \sum_{j=1}^{n^2+n} a_{\lceil xn \rceil j}(G_n) = \begin{cases} 1 & \text{for } x \in [0, 1], \\ \dfrac{1}{n} & \text{for } x \in (1, n+1]. \end{cases}$$

This converges to the function $d(x) = \mathbf{1}\{x \in [0, 1]\}$ in $L_1([0, K])$. To see this, fixing $K \geq 1$, note that $\int_0^K |d_{W_{G_n}}(u) - d(u)| \,\mathrm{d}u = \frac{K-1}{n} \to 0$. Therefore, the conditions of Theorem 1.1 hold with $\lambda_0 = 0$, $W = 0$, and $d(x) = \mathbf{1}\{x \in [0, 1]\}$ (by the discussion in Remark 1.2 and Observation 3.2). Hence,

$$T_n \xrightarrow{D} \mathrm{Pois}(N) \quad \text{where } N \sim \mathrm{Pois}(1).$$

This is a type of *compound Poisson distribution*: a special case of the Poisson mixture, where the mean itself is a Poisson random variable (recall (1.4) with $\lambda = 1$).

One can easily modify the example above to construct graph sequences for which the quadratic component and the Poisson mixture component appear together in the limit:

EXAMPLE 5 (Coexistence I). Let $X_1, X_2, \dots, X_n$ be i.i.d. $\mathrm{Ber}(1/n)$. Construct the graph G_n as follows (see Figure 1):

- Consider disjoint union of n isomorphic copies of the n-star $K_{1,n}^{(1)}, \dots, K_{1,n}^{(n)}$, with vertices labeled as in Example 4 above.
- Place a complete graph K_n on the vertices labeled $1, 2, \dots, n$.
- Place a path of length n^2 with vertices labelled $n^2 + n + 1, \dots, 2n^2 + n$, disjoint from everything else.

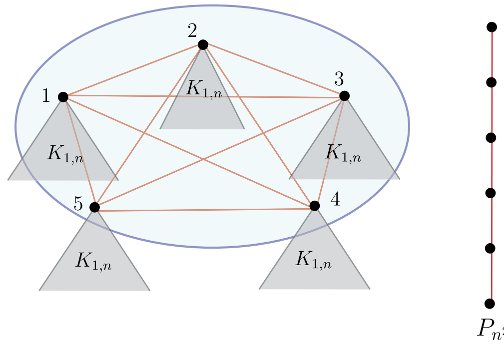

FIG. 1. *Illustration for Example 5.*

Here, $|V(G_n)| = 2n^2 + n$ and $|E(G_n)| = \binom{n}{2} + n^2 + n^2 - 1 = (1 + o(1))\frac{5}{2}n^2$, and, hence, (1.6) holds. Then by arguments similar to Example 4 above, it is easy to check that the conditions of Theorem 1.1 hold with

$$W(x, y) = \mathbf{1}\{(x, y) \in [0, 1]^2\}, \qquad d(x) = 2 \cdot \mathbf{1}\{x \in [0, 1]\}, \quad \text{and} \quad \lambda_0 = 1. \tag{2.10}$$

Then $\Delta(x) = d(x) - \int_0^\infty W(x, y)\,dy = \mathbf{1}\{x \in [0, 1]\}$, and by Theorem 1.1,

$$T_n \xrightarrow{D} Q_1 + Q_2 + Q_3, \tag{2.11}$$

where $Q_3 \sim N_1 \sim \text{Pois}(1)$ is independent of (Q_1, Q_2), and the joint moment generating function of (Q_1, Q_2) is:

$$\mathbb{E}\exp\{-t_1 Q_1 - t_2 Q_2\} = \mathbb{E}\exp\left\{-t_1\binom{N_2}{2} - (1 - e^{-t_2})N_2\right\}, \tag{2.12}$$

where $N_2 \sim \text{Pois}(1)$. In other words, with a slight abuse of notation, we can write

$$T_n \xrightarrow{D} \binom{N_2}{2} + \text{Pois}(N_2) + N_1,$$

where N_1 is independent of $\binom{N_2}{2} + \text{Pois}(N_2)$.

By repeating the constructions above, it is possible to have distributions where the range of the integrals in (1.14) are infinite (unlike in the example above, where the range of the integral reduces to $[0, 1]$ because of (2.10)):

EXAMPLE 6 (Coexistence II). Suppose $X_1, X_2, \ldots, X_n$ be i.i.d. Ber$(1/n)$. For $s \geq 1$, let $a_s = \frac{1}{16^s}$, $b_s = 4^s$, and $c_s = \frac{1}{32^s}$. Now, construct the graph G_n as follows:

- For each $s \in [\lceil \log_4 n \rceil]$, take $\lceil b_s n \rceil$ disjoint isomorphic copies of the $\lceil a_s n \rceil$-stars $K^{(1)}_{1,\lceil a_s n \rceil}$, $K^{(2)}_{1,\lceil a_s n \rceil}, \ldots, K^{(\lceil b_s n \rceil)}_{1,\lceil a_s n \rceil}$. Label the central vertices of the $\lceil a_s n \rceil$-stars, $\lceil b_{s-1} n \rceil + 1, \ldots, \lceil b_s n \rceil$, where $b_0 = 0$.
- For each $s \in [\lceil \log_4 n \rceil]$, place an Erdős–Rényi random graph $G(\lceil b_s n \rceil, c_s)$ on the vertices labeled $\lceil b_{s-1} n \rceil + 1, \ldots, \lceil b_s n \rceil$.

Note that $|V(G_n)| = \sum_{s=1}^{\lceil \log_4 n \rceil} \lceil b_s n \rceil (\lceil a_s n \rceil + 1) = \Theta(n^2)$, and

$$\mathbb{E}|E(G_n)| = (1 + o(1))\left(\frac{1}{2}\sum_{s=1}^{\lceil \log_4 n \rceil} \lceil b_s n \rceil^2 c_s + \sum_{s=1}^{\lceil \log_4 n \rceil} \lceil b_s n \rceil \lceil a_s n \rceil\right) = \Theta(n^2).$$

(Note that the choice $p_n = 1/n$ implies (1.6) holds.) As before, it can be verified that the conditions of Theorem 1.1 hold with $\lambda_0 = 0$,

$$W(x, y) = \begin{cases} c_s & \text{for } x, y \in [r_{s-1}, r_s], \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}$$

$$d(x) = \begin{cases} c_s + a_s & \text{for } x \in [r_{s-1}, r_s], \\ 0 & \text{otherwise,} \end{cases} \tag{2.13}$$

for $s \geq 1$ and $r_s = \sum_{i=0}^{s} b_i$ is the sth partial sum of the sequence $\{b_i\}_{i \geq 1}$. Now, for $t_1 \geq 0$ and recalling $\phi_{W,t_1}(x, y) := \log(1 - W(x, y) + W(x, y)e^{-t_1})$, it follows from Example 8 that,

$$\int_0^\infty \int_0^\infty \phi_{W,t_1}(x, y)\, \mathrm{d}N(x)\, \mathrm{d}N(y) = \sum_{s=1}^{\infty} \psi_{W,t_1}(s) \binom{N_s}{2}, \tag{2.14}$$

where $\psi_{W,t_1}(s) := \log(1 - c_s + c_s e^{-t_1})$, for $s \geq 1$, and $N_s \sim \mathrm{Pois}(b_s)$ and $\{N_1, N_2, \ldots\}$ are independent. Moreover,

$$\int_0^\infty \Delta(x)\, \mathrm{d}N(x) = \int_0^\infty \left(d(x) - \int_0^\infty W(x, y)\, \mathrm{d}y \right) \mathrm{d}N(x) = \sum_{s=1}^{\infty} a_s N_s. \tag{2.15}$$

Combining (2.14) and (2.15), with Theorem 1.1, it follows that $T_n \xrightarrow{D} Q_1 + Q_2$, where

$$\mathbb{E}\exp\{-t_1 Q_1 - t_2 Q_2\} = \mathbb{E}\exp\left\{ -t_1 \sum_{s=1}^{\infty} \psi_{W,t_1}(c_s) \binom{N_s}{2} - (1 - e^{-t_2}) \sum_{s=1}^{\infty} a_s N_s \right\}. \tag{2.16}$$

This can be rewritten, by comparing moment generating functions, as

$$T_n \xrightarrow{D} \sum_{s=1}^{\infty} \mathrm{Bin}\left(\binom{N_s}{2}, \frac{1}{32^s} \right) + \mathrm{Pois}\left(\sum_{s=1}^{\infty} \frac{N_s}{16^s} \right),$$

where, as before, $N_s \sim \mathrm{Pois}(4^s)$ are independent, and conditional on the sequence $\{N_1, N_2, \ldots\}$, the Poisson and the Binomials above are independent.

We conclude with an example where T_n does not have a limit in distribution, showing the necessity of the conditions in Theorem 1.1.

EXAMPLE 7 (Nonexistence of limit). Let $X_1, X_2, \ldots, X_n$ be i.i.d. $\mathrm{Ber}(1/n)$. We will construct a graph sequence $\{G_n\}_{n \geq 1}$ for which T_n does not converge in dsitribution. Let G_n be defined as:

- Consider a Erdős–Rényi random graph $G(n, \frac{1}{4})$ on the vertices labelled $1, 2, \ldots, n$, and another independent Erdős–Rényi random graph $G(n, \frac{1}{2})$ on the vertices labelled $n+1, n+2, \ldots, 2n$.
- For n odd, attach n disjoint n-stars $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(n)}$, with central vertices at $1, 2, \ldots, n$ respectively.
- For n even, attach n disjoint n-stars $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(n)}$, with central vertices at $n+1, n+2, \ldots, 2n$, respectively.

Here, $|V(G_n)| = \Theta(n^2)$ and $\mathbb{E}|E(G_n)| = \Theta(n^2)$, hence, (1.6) holds. Now, from the arguments in (2.6) and Example 4, it follows that the contribution to T_n from the $G(n, \frac{1}{4})$ and $G(n, \frac{1}{2})$ components converge to $\mathrm{Bin}(\binom{N_1}{2}, \frac{1}{4})$ and $\mathrm{Bin}(\binom{N_2}{2}, \frac{1}{2})$, respectively, where N_1, N_2 are independent Pois(1). Moreover, the contribution of the n disjoint stars converge to

$\mathrm{Pois}(N_1)$ along the odd subsequence and $\mathrm{Pois}(N_2)$ along the even subsequence. Therefore, along the odd subsequence,

$$T_n \xrightarrow{D} \mathrm{Bin}\left(\binom{N_1}{2}, \frac{1}{4}\right) + \mathrm{Pois}(N_1) + \mathrm{Bin}\left(\binom{N_2}{2}, \frac{1}{2}\right), \tag{2.17}$$

and along the even subsequence

$$T_n \xrightarrow{D} \mathrm{Bin}\left(\binom{N_1}{2}, \frac{1}{4}\right) + \mathrm{Pois}(N_2) + \mathrm{Bin}\left(\binom{N_2}{2}, \frac{1}{2}\right), \tag{2.18}$$

where N_1, N_2 are independent Pois(1), and the conditional on N_1, N_2, the Poisson and the two binomials are independent. Clearly, the distributions in (2.17) and (2.18) are not the same (this can be easily seen by computing their second moments), that is, T_n does not converge in distribution. This is because, for all $K \geq 1$, the function $d_{W_{G_n}}$ converges in $L_1([0, K])$, to the function $d_+(x) = \mathbf{1}\{x \in [0, 1]\}$ along the odd subsequence, and to the function $d_-(x) = \mathbf{1}\{x \in [1, 2]\}$ along the even subsequence, respectively. This shows condition (1.12) in Theorem 1.1 does not hold. In fact, in this case it can be shown that there is no permutation of the vertices $\{1, 2, \ldots, 2n\}$ for which conditions (1.11) and (1.12) simultaneously hold, in the permuted graph (recall the discussion in Remark 1.2).

3. Proof of Theorem 1.1. For positive integers $a < b$, denote by $[a, b] := \{a, a + 1, \ldots, b\}$. (We will often slightly abuse notation and also use $[a, b]$ to denote the closed interval with points $a, b \in \mathbb{R}$, whenever it is clear from the context.) Throughout we assume that the vertices of G_n are labelled $\{1, 2, \ldots, n\}$ in nonincreasing order of the degrees. Recall that d_v denotes the degree of the vertex labelled v.

OBSERVATION 3.1. *If the vertices of G_n are labelled $\{1, 2, \ldots, n\}$ in the nonincreasing order of the degrees $d_1 \geq d_2 \geq \cdots \geq d_n$, then*

$$\lim_{K\to\infty} \lim_{n\to\infty} \frac{d_{\lceil Kr_n\rceil}}{r_n} = 0. \tag{3.1}$$

PROOF. Note that

$$2\left|E(G_n)\right| = \sum_{v=1}^{n} d_v \geq \sum_{v=1}^{\lceil Kr_n\rceil} d_v \geq \lceil Kr_n\rceil d_{\lceil Kr_n\rceil},$$

which implies, by (1.6), $d_{\lceil Kr_n\rceil} \lesssim \frac{r_n}{K}$, hence (3.1) holds. $\square$

The first step in the proof of Theorem 1.1 is a truncation argument, which shows that vertices with "large" degree have negligible contribution to T_n. To this end, recall the definition of $T_{n,M}$ from (1.17). We begin by showing that the difference between T_n and the truncation $T_{n,M}$ above, goes to zero in probability.

LEMMA 3.1. *Let T_n and $T_{n,M}$ be as in defined in (1.5) and (1.17), respectively. Then*

$$\lim_{M\to\infty} \lim_{n\to\infty} \mathbb{P}(T_n \neq T_{n,M}) = 0.$$

PROOF. Fix $n \geq 1$ and $M > 1$. Then

$$\begin{aligned}\mathbb{P}(T_n \neq T_{n,M}) &\leq \mathbb{P}\big(\exists a \in V(G_n) : d_a > Mr_n \text{ and } X_a = 1\big)\\ &\leq \sum_{a\in V(G_n):d_a>Mr_n} \mathbb{P}(X_a = 1)\end{aligned}$$

(recall $r_n = 1/p_n$).
$$= \sum_{a \in V(G_n)} p_n \mathbf{1}\{d_a > M/p_n\}$$
$$\leq \sum_{a \in V(G_n)} \frac{p_n^2 d_a}{M} = \frac{2|E(G_n)| p_n^2}{M},$$

which goes to zero under the double limit, by assumption (1.6). □

This shows that it suffices to derive the limiting distribution of $T_{n,M}$. Now, fix $K \geq 1$, and define

$$V^+_{G_n,K} := [\lceil K r_n \rceil] \quad \text{and} \quad V^-_{G_n,K} := [\lceil K r_n \rceil + 1, n],$$

the first $\lceil K r_n \rceil$ vertices and the last $n - \lceil K r_n \rceil$ vertices, respectively. Denote by

$$G^+_{n,K} := G_n[V^+_{G_n,K}] \quad \text{and} \quad G^-_{n,K} := G_n[V^-_{G_n,K}],$$

the subgraphs of G_n induced by $V^+_{G_n,K}$ and $V^-_{G_n,K}$, respectively, where for $S \subseteq V(G_n)$, $G_n[S]$ denotes induced sub-graph of G_n with vertex set S. Finally, let $G^{\pm}_{n,K}$ be the subgraph of G_n formed by the union of edges with one end point in $V^+_{G_n,K}$ and the other in $V^-_{G_n,K}$. Note that by definition the subgraphs $G^+_{n,K}$, $G^{\pm}_{n,K}$, and $G^-_{n,K}$ partition the edges of G_n, that is, $E(G_n) = E(G^+_{n,K}) \cup E(G^{\pm}_{n,K}) \cup E(G^-_{n,K})$ is a disjoint partition of $E(G_n)$. Therefore, we can decompose $T_{n,M}$ as follows:

$$T_{n,M} = \sum_{(u,v) \in E(G_n)} X_{u,M} X_{v,M} = T^+_{n,K,M} + T^{\pm}_{n,K,M} + T^-_{n,K,M}, \tag{3.2}$$

where

$$T^+_{n,K,M} := \sum_{(u,v) \in E(G^+_{n,K})} X_{u,M} X_{v,M} \quad \text{and} \quad T^-_{n,K,M} := \sum_{(u,v) \in E(G^-_{n,K})} X_{u,M} X_{v,M}, \tag{3.3}$$

and

$$T^{\pm}_{n,K,M} := \sum_{u \in V^+_{G_n,K}} \sum_{v \in V^-_{G_n,K}} a_{uv}(G_n) X_{u,M} X_{v,M}. \tag{3.4}$$

Hereafter, we will refer to $T^+_{n,K,M}$, $T^{\pm}_{n,K,M}$, and $T^-_{n,K,M}$ as the *high-degree component*, the *intermediate component*, and the *low-degree component* of $T_{n,M}$, respectively.

The proof of Theorem 1.1 involves deriving the joint distribution of the three terms in (3.2). It has the following main steps:

(1) We begin by showing that the moments of low-degree component $T^-_{n,K,M}$ are asymptotically equal to the moments of the random variable $W^-_{n,K} := \sum_{(u,v) \in E(G^-_{n,K})} R_{uv}$, where $\{R_{uv}\}_{(u,v) \in E(G^-_{n,K})}$ is a collection of independent Bernoulli(p_n^2) random variables (see Lemma 3.2 on Section 3.1). For each $a \geq 1$ fixed, the proof involves expressing the ath moment of $T^-_{n,K,M}$ (and similarly for $W^-_{n,K}$) as a sum over subgraphs of $G^-_{n,K}$ with at most a edges and then estimating the number of copies of every (fixed) subgraph in $G^-_{n,K}$ using the bound on the maximum degree of $G^-_{n,K}$. Note that, since $W^-_{n,K}$ is the sum of independent Bernoulli random variables, its limiting distribution can be easily calculated, a fact we will leverage later.

(2) Next, we show that the low-degree component $T^-_{n,K,M}$ and the joint distribution of the intermediate and high-degree components $(T^+_{n,K,M}, T^{\pm}_{n,K,M})$ are asymptotically independent in moments (see Lemma 3.3 in Section 3.2). As in the previous case, the proof proceeds by expressing the joint mixed moments in terms of certain subgraph counts and estimating them using appropiate degree bounds.

(3) The next step is to show that the moments of intermediate component $T^{\pm}_{n,K,M}$ are asymptotically equal to the moments of $Z_{n,K,M} = \sum_{u \in V^+_{G_n,K}} \sum_{v \in V^-_{G_n,K}} a_{uv}(G_n) J_{uv} X_{u,M}$, where $\{J_{uv}\}_{(u,v)\in E(G^{\pm}_{n,K})}$ is a collection of independent Bernoulli(p_n) (see Lemma 3.4 in Section 3.3). This will allow us to transfer from the distribution of $T^{\pm}_{n,K,M}$ to that of $Z_{n,K,M}$, which has more independence structure than $T^{\pm}_{n,K,M}$. By combining this with the results from the previous two steps, we can then show that the joint mixed moments of $(T^+_{n,K,M}, T^{\pm}_{n,K,M}, T^-_{n,K,M})$ are asymptotically equal to joint mixed moments of $(T^+_{n,K,M}, Z_{n,K,M}, W^-_{n,K})$ (Proposition 3.1).

(4) Having transferred in moments from $(T^+_{n,K,M}, T^{\pm}_{n,K,M}, T^-_{n,K,M})$ to $(T^+_{n,K,M}, Z_{n,K,M}, W^-_{n,K})$, we proceed to derive the limiting distribution of the latter. To this end, the first step is to show that under the assumptions of Theorem 1.1, the joint mixed moments of $(T^+_{n,K,M}, Z_{n,K,M})$ converges, as $n \to \infty$ and for all fixed K, M large enough. We then show that this convergence is also in distribution, by verifying a bivariate Carleman moment condition (for the Stieltjes moment problem) [1, 19, 28]. The details are given in Section 3.4.

(5) Next, we proceed to compute the joint distribution of $(T^+_{n,K,M}, Z_{n,K,M})$. To this end, we replace the graph in $G^+_{n,K}$ by an inhomogeneous random graph which has the same graph limit as $G^+_{n,K}$ (Section 3.5). In this case, the limiting moment generating function can be explicitly computed by first taking the expectation with respect to the randomness of the graph. The existence of the limit proved in the previous step can then be used to show that this has the same limit as $(T^+_{n,K,M}, Z_{n,K,M})$.

(6) The proof of (1.13) is completed in Section 3.6, which entails moving from the joint distribution of $(T^+_{n,K,M}, Z_{n,K,M}, W^-_{n,K})$ to that of the actual variables $(T^+_{n,K,M}, T^{\pm}_{n,K,M}, T^-_{n,K,M})$, by verifying another Carleman moment condition and taking limits in the various parameters.

3.1. *Moment approximation for $T^-_{n,K,M}$.* Define

$$W^-_{n,K} := \sum_{(u,v)\in E(G^-_{n,K})} R_{uv}, \tag{3.5}$$

where $\{R_{uv}\}_{(u,v)\in E(G^-_{n,K})}$ is a collection of independent Bernoulli(p_n^2) random variables. In the following lemma, we show that $T^-_{n,K,M}$ and $W^-_{n,K}$ are close in moments. To this end, we need a few notations: For any two graphs H and G, let $N(H,G)$ denote the number of isomorphic copies of H in G.

LEMMA 3.2. *Fix $M \geq 1$. Then, under the assumptions of Theorem 1.1, for every positive integer $a \geq 1$,*

$$\lim_{K\to\infty} \limsup_{n\to\infty} \left|\mathbb{E}\left[(T^-_{n,K,M})^a\right] - \mathbb{E}\left[(W^-_{n,K})^a\right]\right| = 0.$$

Moreover, $\limsup_{K\to\infty} \limsup_{n\to\infty} \mathbb{E}[(T^-_{n,K,M})^a] < C(a)$, *for some constant $C(a) > 0$.*

PROOF. Fix $\varepsilon \in (0,1)$. Choose n, K large enough so that $\max_{v\in V^-_{G_n,K}} d_v = d_{\lceil K r_n\rceil+1} \leq \varepsilon r_n$, which can be done by Observation 3.1. This implies $X_{v,M} = X_v$, for all $v \in V^-_{G_n,K}$, when

n, K are large enough, and by the the multinomial expansion,

$$\mathbb{E}\big[(T^-_{n,K,M})^a\big] = \sum_{(u_1,v_1)\in E(G^-_{n,K})} \sum_{(u_2,v_2)\in E(G^-_{n,K})} \cdots \sum_{(u_a,v_a)\in E(G^-_{n,K})} \mathbb{E}\Bigg[\prod_{s=1}^{a} X_{u_s} X_{v_s}\Bigg], \tag{3.6}$$

$$\mathbb{E}\big[(W^-_{n,K})^a\big] = \sum_{(u_1,v_1)\in E(G^-_{n,K})} \sum_{(u_2,v_2)\in E(G^-_{n,K})} \cdots \sum_{(u_a,v_a)\in E(G^-_{n,K})} \mathbb{E}\Bigg[\prod_{s=1}^{a} R_{u_s v_s}\Bigg]. \tag{3.7}$$

Now, let H be the graph formed by the union of the edges $(u_1, v_1), (u_2, v_2), \dots, (u_a, v_a)$. Then

$$\mathbb{E}\Bigg[\prod_{s=1}^{a} X_{u_s} X_{v_s}\Bigg] = p_n^{|V(H)|} \quad \text{and} \quad \mathbb{E}\Bigg[\prod_{s=1}^{a} R_{u_s v_s}\Bigg] = p_n^{2|E(H)|}. \tag{3.8}$$

If $\mathcal{H}_a$ denotes the set of all nonisomorphic graphs with at most a edges and no isolated vertex, then (3.6), (3.7), and (3.8) combined gives

$$\begin{aligned} \big|\mathbb{E}\big[(T^-_{n,K,M})^a\big] - \mathbb{E}\big[(W^-_{n,K})^a\big]\big| &\lesssim_a \sum_{H\in\mathcal{H}_a} N(H, G^-_{n,K})\big|p_n^{|V(H)|} - p_n^{2|E(H)|}\big| \\ &= \sum_{\substack{H\in\mathcal{H}_a \\ |V(H)|<2|E(H)|}} N(H, G^-_{n,K})\big|p_n^{|V(H)|} - p_n^{2|E(H)|}\big| \\ &\leq \sum_{\substack{H\in\mathcal{H}_a \\ |V(H)|<2|E(H)|}} N(H, G^-_{n,K}) p_n^{|V(H)|}, \end{aligned} \tag{3.9}$$

where the second and third steps use the fact that $|V(H)| \leq 2|E(H)|$, since graphs in $\mathcal{H}_a$ have no isolated vertex.

Now, for any connected graph $H \in \mathcal{H}_a$,

$$\begin{aligned} N(H, G^-_{n,K}) &\lesssim_a \big|E(G^-_{n,K})\big|\Big(\max_{v\in V_{G^-_{n,K}}} d_v\Big)^{|V(H)|-2} \\ \text{(by Observation (3.1))} \qquad &\leq \big|E(G^-_{n,K})\big|\varepsilon^{|V(H)|-2} r_n^{|V(H)|-2} \\ &\lesssim \varepsilon^{|V(H)|-2} r_n^{|V(H)|}, \end{aligned}$$

where the last step uses $|E(G^-_{n,K})| \lesssim r_n^2$ by (1.6).

Therefore, if $H \in \mathcal{H}_a$ has $\nu(H)$ connected components, then using the above bound separately on each of the connected components gives,

$$N(H, G^-_{n,K}) \lesssim_a \varepsilon^{|V(H)|-2\nu(H)} p_n^{-|V(H)|}. \tag{3.10}$$

Using the estimate above and (3.9) gives,

$$\big|\mathbb{E}\big[(T^-_{n,K,M})^a\big] - \mathbb{E}\big[(W^-_{n,K})^a\big]\big| \lesssim_a \sum_{\substack{H\in\mathcal{H}_a \\ |V(H)|<2|E(H)|}} \varepsilon^{|V(H)|-2\nu(H)}. \tag{3.11}$$

Now, suppose $H \in \mathcal{H}_a$ is such that $|V(H)| < 2|E(H)|$. Note that all connected components of H contain at least two vertices, and at least one connected component of H must contain at least three vertices (otherwise H is a disjoint union of edges, and $|V(H)| = 2|E(H)|$). This implies, $|V(H)| > 2\nu(H)$, where $\nu(H)$ is the number of connected components of H. Since $\varepsilon > 0$ is arbitrary and cardinality of the set $\mathcal{H}_a$ is fixed (free of n), the RHS of (3.11) can be made arbitrarily small, and so the LHS of (3.11) converges to 0 under the double limit of n goes to infinity followed by K goes to infinity, which is the first desired result.

Finally, from (3.6), (3.8), and (3.10) we have

$$\mathbb{E}[(T^-_{n,K,M})^a] \lesssim_a \sum_{H\in\mathcal{H}_a} N(H, G^-_{n,K}) p_n^{|V(H)|} \lesssim_a \sum_{H\in\mathcal{H}_a} \varepsilon^{|V(H)|-2\nu(H)} \lesssim_a 1,$$

because, as before, the sum is over a finite index set free of n. □

3.2. *Independence in moments of* $(T^+_{n,K,M}, T^{\pm}_{n,K,M})$ *and* $T^-_{n,K,M}$. In this section, we will show that the mixed moments $(T^+_{n,K,M}, T^{\pm}_{n,K,M})$ and $T^-_{n,K,M}$ factorize in the limit.

LEMMA 3.3. *Fix nonnegative integers* a, b, c, *and* $M > 0$. *Then under the assumptions of Theorem* 1.1,

$$\lim_{K\to\infty} \limsup_{n\to\infty} \big|\mathbb{E}[(T^+_{n,K,M})^a (T^{\pm}_{n,K,M})^b (T^-_{n,K,M})^c] - \mathbb{E}[(T^+_{n,K,M})^a (T^{\pm}_{n,K,M})^b]\mathbb{E}[(T^-_{n,K,M})^c]\big| = 0.$$

PROOF. Note that there is nothing to prove if $c = 0$. Moreover, since $T^+_{n,K,M}$ and $T^-_{n,K,M}$ are independent for each n and K (they are defined on disjoint sets of vertices of G_n), the case $b = 0$ follows trivially. Therefore, we assume b and c are both positive.

Let $\mathcal{E}^{a,b,c}_{n,K}$ be the collection of $(a+b+c)$-tuples of the form

$$\boldsymbol{e} = \big((u_1, v_1), \dots, (u_a, v_a), (u'_1, v'_1), \dots, (u'_b, v'_b), (u''_1, v''_1), \dots, (u''_c, v''_c)\big),$$

where $(u_1, v_1), \dots, (u_a, v_a) \in E(G^+_{n,K})$, $(u'_1, v'_1), \dots, (u'_b, v'_b) \in E(G^{\pm}_{n,K})$ (with $u'_1, \dots, u'_b \in V^+_{G_n,K}$ and $v'_1, \dots, v'_b \in V^-_{G_n,K}$), and $(u''_1, v''_1), \dots, (u''_c, v''_c) \in E(G^-_{n,K})$. Further, define $\mathcal{D}^{a,b,c}_{n,K}$ as the set of all $\boldsymbol{e} \in \mathcal{E}^{a,b,c}_{n,K}$ such that the sets $\{v'_1, \dots, v'_b\}$ and $\{u''_1, v''_1, \dots, u''_a, v''_a\}$ are disjoint. Then

$$\begin{aligned}
&\mathbb{E}[(T^+_{n,K,M})^a (T^{\pm}_{n,K,M})^b (T^-_{n,K,M})^c] \\
&\quad= \sum_{\boldsymbol{e}\in\mathcal{E}^{a,b,c}_{n,K}} \mathbb{E}\Bigg[\prod_{s=1}^a X_{u_s,M}X_{v_s,M} \prod_{s=1}^b X_{u'_s,M}X_{v'_s,M} \prod_{s=1}^c X_{u''_s,M}X_{v''_s,M}\Bigg] \\
&\quad= \sum_{\boldsymbol{e}\in\mathcal{D}^{a,b,c}_{n,K}} \mathbb{E}\Bigg[\prod_{s=1}^a X_{u_s,M}X_{v_s,M} \prod_{s=1}^b X_{u'_s,M}X_{v'_s,M}\Bigg]\mathbb{E}\Bigg[\prod_{s=1}^c X_{u''_s,M}X_{v''_s,M}\Bigg] \qquad (3.12)\\
&\qquad+ \sum_{\boldsymbol{e}\in\mathcal{E}^{a,b,c}_{n,K}\setminus\mathcal{D}^{a,b,c}_{n,K}} \mathbb{E}\Bigg[\prod_{s=1}^a X_{u_s,M}X_{v_s,M} \prod_{s=1}^b X_{u'_s,M}X_{v'_s,M} \prod_{s=1}^c X_{u''_s,M}X_{v''_s,M}\Bigg].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}[(T^+_{n,K,M})^a (T^{\pm}_{n,K,M})^b]\mathbb{E}[(T^-_{n,K,M})^c] \\
&\quad= \sum_{\boldsymbol{e}\in\mathcal{E}^{a,b,c}_{n,K}} \mathbb{E}\Bigg[\prod_{s=1}^a X_{u_s,M}X_{v_s,M} \prod_{s=1}^b X_{u'_s,M}X_{v'_s,M}\Bigg]\mathbb{E}\Bigg[\prod_{s=1}^c X_{u''_s,M}X_{v''_s,M}\Bigg] \qquad (3.13)
\end{aligned}$$

$$= \sum_{\boldsymbol{e} \in \mathcal{D}_{n,K}^{a,b,c}} \mathbb{E}\left[\prod_{s=1}^{a} X_{u_s,M} X_{v_s,M} \prod_{s=1}^{b} X_{u'_s,M} X_{v'_s,M}\right] \mathbb{E}\left[\prod_{s=1}^{c} X_{u''_s,M} X_{v''_s,M}\right]$$

$$+ \sum_{\boldsymbol{e} \in \mathcal{E}_{n,K}^{a,b,c} \setminus \mathcal{D}_{n,K}^{a,b,c}} \mathbb{E}\left[\prod_{s=1}^{a} X_{u_s,M} X_{v_s,M} \prod_{s=1}^{b} X_{u'_s,M} X_{v'_s,M}\right] \mathbb{E}\left[\prod_{s=1}^{c} X_{u''_s,M} X_{v''_s,M}\right].$$

By taking the difference of (3.12) and (3.13) it follows that, in order to prove the lemma, it suffices to show the following two statements:

$$\sum_{\boldsymbol{e} \in \mathcal{E}_{n,K}^{a,b,c} \setminus \mathcal{D}_{n,K}^{a,b,c}} \mathbb{E}\left[\prod_{s=1}^{a} X_{u_s,M} X_{v_s,M} \prod_{s=1}^{b} X_{u'_s,M} X_{v'_s,M} \prod_{s=1}^{c} X_{u''_s,M} X_{v''_s,M}\right] \to 0, \tag{3.14}$$

as $n \to \infty$ followed by $K \to \infty$, and

$$\sum_{\boldsymbol{e} \in \mathcal{E}_{n,K}^{a,b,c} \setminus \mathcal{D}_{n,K}^{a,b,c}} \mathbb{E}\left[\prod_{s=1}^{a} X_{u_s,M} X_{v_s,M} \prod_{s=1}^{b} X_{u'_s,M} X_{v'_s,M}\right] \mathbb{E}\left[\prod_{s=1}^{c} X_{u''_s,M} X_{v''_s,M}\right] \to 0, \tag{3.15}$$

as $n \to \infty$ followed by $K \to \infty$.

To this end, define $\mathcal{E}_{n,K,M}^{a,b,c}$ to be the set of all $\boldsymbol{e} \in \mathcal{E}_{n,K}^{a,b,c}$, such that the following three conditions hold: (1) $\max\{d_{u_s}, d_{v_s}\} \le M r_n$, for all $s \in [a]$, (2) $\max\{d_{u'_s}, d_{v'_s}\} \le M r_n$, for all $s \in [b]$, and (3) $\max\{d''_{u_s}, d''_{v_s}\} \le M r_n$, for all $s \in [c]$. Let $\mathcal{D}_{n,K,M}^{a,b,c} = \mathcal{E}_{n,K,M}^{a,b,c} \cap \mathcal{D}_{n,K}^{a,b,c}$. Then, (3.14) becomes:

$$\lim_{K \to \infty} \limsup_{n \to \infty} \sum_{\boldsymbol{e} \in \mathcal{E}_{n,K,M}^{a,b,c} \setminus \mathcal{D}_{n,K,M}^{a,b,c}} \mathbb{E}\left[\prod_{s=1}^{a} X_{u_s} X_{v_s} \prod_{s=1}^{b} X_{u'_s} X_{v'_s} \prod_{s=1}^{c} X_{u''_s} X_{v''_s}\right] = 0. \tag{3.16}$$

If H is the graph formed by the union of the edges $(u_1, v_1), \ldots, (u_a, v_a)$, $(u'_1, v'_1), \ldots, (u'_b, v'_b)$, $(u''_1, v''_1), \ldots, (u''_c, v''_c)$, then

$$\mathbb{E}\left[\prod_{s=1}^{a} X_{u_s} X_{v_s} \prod_{s=1}^{b} X_{u'_s} X_{v'_s} \prod_{s=1}^{c} X_{u''_s} X_{v''_s}\right] = p_n^{|V(H)|}.$$

Note that this graph H must have at least two edges (u'_i, v'_i) and (u''_j, v''_j), such that $v'_i = u''_j$ or $v'_i = v''_j$, since for any $\boldsymbol{e} \in \mathcal{E}_{n,K,M}^{a,b,c} \setminus \mathcal{D}_{n,K,M}^{a,b,c}$, the set $\{v'_1, \ldots, v'_b\}$ intersect the set $\{u''_1, v''_1, \ldots, u''_a, v''_a\}$, that is, H has a two-star $K_{1,2}$, with central vertex in $V_{G_n,K}^{-}$. Let $V_M := \{v \in V(G_n) : d_v \le M r_n\}$ and $N_{a,b,c}(H, G_n[V_M])$ be the number ways of forming a graph isomorphic to H, with a edges from $G_{n,K}^{+}[V_M]$, b edges from $G_{n,K}^{\pm}[V_M]$, and c edges from $G_{n,K}^{-}[V_M]$, such that the resulting graph contains a $K_{1,2}$, with central vertex in $V_{G_n,K}^{-}$ (and one edge in $E(G_{n,K}^{-}[V_M])$ and the other in $E(G_{n,K}^{\pm}[V_M])$). Then

$$\begin{aligned} &\sum_{\boldsymbol{e} \in \mathcal{E}_{n,K,M}^{a,b,c} \setminus \mathcal{D}_{n,K,M}^{a,b,c}} \mathbb{E}\left[\prod_{s=1}^{a} X_{u_s} X_{v_s} \prod_{s=1}^{b} X_{u'_s} X_{v'_s} \prod_{s=1}^{c} X_{u''_s} X_{v''_s}\right] \\ &\quad \lesssim \sum_{H \in \mathcal{H}_{a,b,c}} N_{a,b,c}(H, G_n[V_M]) p_n^{|V(H)|}, \end{aligned} \tag{3.17}$$

where $\mathcal{H}_{a,b,c}$ is the set of all nonisomorphic graphs with at most $2(a+b+c)$ vertices, none of which is isolated, which contains at least one $K_{1,2}$ (the two-star) as a subgraph.

Now, we proceed to bound $N_{a,b,c}(H, G_n[V_M])$: Note that for any connected $F \in \mathcal{H}_{a,b,c}$,

$$N_{a,b,c}\big(F, G_n[V_M]\big) \lesssim_{a,b,c} \big|E\big(G_n[V_M]\big)\big|(Mr_n)^{|V(F)|-2} \leq \big|E(G_n)\big|(Mr_n)^{|V(F)|-2} \lesssim_{M,F} r_n^{|V(F)|}, \tag{3.18}$$

using (1.6). Now, suppose $H \in \mathcal{H}_{a,b,c}$ has connected components $H_1, H_2, \dots, H_{\nu(H)}$, and without loss of generality, assume H_1 has a two-star $K_{1,2}$, with central vertex in $V^-_{G_n,K}$ (and one edge in $E(G^-_{n,K}[V_M])$ and the other in $E(G^{\pm}_{n,K}[V_M])$). Therefore, choosing this two-star in at most $|E(G_n[V_M])| \cdot \max\{d_v : v \in V^-_{G_n,K}\}$ ways and each of the remaining $|V(H_1)| - 3$ vertices in at most Mr_n ways gives the bound

$$\begin{aligned} N_{a,b,c}\big(H_1, G_n[V_M]\big) &\lesssim_{a,b,c} \big|E\big(G_n[V_M]\big)\big|\Big(\max_{v\in V^-_{G_n,K}} d_v\Big)(Mr_n)^{(|V(H_1)|-3)} \\ &\lesssim_{a,b,c,M} \big|E(G_n)\big| r_n^{(|V(H_1)|-3)}\Big(\max_{v\in V^-_{G_n,K}} d_v\Big) \\ &\lesssim_{a,b,c,M} r_n^{(|V(H_1)|-1)} d_{\lceil Kr_n\rceil+1}, \end{aligned} \tag{3.19}$$

using (1.6). Now, combining (3.18) and (3.19) gives

$$\begin{aligned} N_{a,b,c}\big(H, G_n[V_M]\big) &\leq \prod_{j=1}^{\nu(H)} N_{a,b,c}\big(H_j, G_n[V_M]\big) \\ &\lesssim_{a,b,c,M,H} r_n^{(|V(H)|-1)} d_{\lceil Kr_n\rceil+1}. \end{aligned} \tag{3.20}$$

This implies

$$\lim_{K\to\infty} \lim_{n\to\infty} \frac{1}{r_n^{|V(H)|}} N_{a,b,c}\big(H, G_n[V_M]\big) = 0,$$

by Observation 3.1. Thus (3.16) follows, because the sum in the right hand side of (3.17) is over a finite set ($|\mathcal{H}_{a,b,c}| \leq 2^{\binom{2(a+b+c)}{2}}$). The limit in (3.15) follows similarly, completing the proof of the lemma. $\square$

3.3. *Moment approximation for $T^{\pm}_{n,K,M}$.* Let $\{J_{uv}\}_{(u,v)\in E(G^{\pm}_{n,K})}$ be a collection of independent Bernoulli(p_n) random variables, independent of the collection $\{X_v\}_{v\in V(G_n)}$. Define

$$Z_{n,K,M} = \sum_{u\in V^+_{G_n,K}} \sum_{v\in V^-_{G_n,K}} a_{uv}(G_n) J_{uv} X_{u,M}. \tag{3.21}$$

LEMMA 3.4. *Fix nonnegative integers a, b, and $M > 0$. Then under the assumptions of Theorem 1.1,*

$$\lim_{K\to\infty} \limsup_{n\to\infty} \big|\mathbb{E}\big[(T^+_{n,K,M})^a (T^{\pm}_{n,K,M})^b\big] - \mathbb{E}\big[(T^+_{n,K,M})^a Z^b_{n,K,M}\big]\big| = 0.$$

Moreover, $\limsup_{K\to\infty} \limsup_{n\to\infty} \mathbb{E}[(T^+_{n,K,M})^a Z^b_{n,K,M}] \lesssim_{a,b,M} 1$.

PROOF. Note that there is nothing to prove if $b = 0$, so we assume that $b > 0$. Let $\mathcal{M}^{a,b}_{n,K}$ the collection of $(a+b)$-tuples of the form

$$\boldsymbol{e} = \big((u_1, v_1), \dots, (u_a, v_a), (u'_1, v'_1), \dots, (u'_b, v'_b)\big),$$

where $(u_1, v_1), \dots, (u_a, v_a) \in E(G^+_{n,K}[V_M])$ and $(u'_1, v'_1), \dots, (u'_b, v'_b) \in E(G^{\pm}_{n,K}[V_M])$ (with $u'_1, \dots, u'_b \in V^+_{G_n,K}$ and $v'_1, \dots, v'_b \in V^-_{G_n,K}$).

$$
\begin{aligned}
&\big|\mathbb{E}\big[(T^+_{n,K,M})^a (T^{\pm}_{n,K,M})^b\big] - \mathbb{E}\big[(T^+_{n,K,M})^a Z^b_{n,K,M}\big]\big| \\
&\quad \leq \sum_{\boldsymbol{e} \in \mathcal{M}^{a,b}_{n,K}} \left| \mathbb{E}\left[\prod_{s=1}^{a} X_{u_s} X_{v_s} \prod_{s=1}^{b} X_{u'_s} X_{v'_s}\right] - \mathbb{E}\left[\prod_{s=1}^{a} X_{u_s} X_{v_s} \prod_{s=1}^{b} X_{u'_s} \prod_{s=1}^{b} J_{u'_s v'_s}\right]\right| \qquad (3.22) \\
&\quad = \sum_{\boldsymbol{e} \in \mathcal{M}^{a,b}_{n,K}} \big| p_n^{|V(H_1 \cup H_2)|} - p_n^{|\{u_1, v_1, u_2, v_2, \dots, u_a, v_a, u'_1, u'_2, \dots, u'_b\}| + |E(H_2)|} \big|,
\end{aligned}
$$

where H_1 is the graph formed by the union of the edges $(u_1, v_1), \dots, (u_a, v_a)$, and H_2 is the graph formed by the union of the edges $(u'_1, v'_1), \dots, (u'_b, v'_b)$.

Note that

$$
\begin{aligned}
&\big|\{u_1, v_1, u_2, v_2, \dots, u_a, v_a, u'_1, u'_2, \dots, u'_b\}\big| + \big|E(H_2)\big| \\
&\quad = \big|V(H_1 \cup H_2)\big| - \big|\{v'_1, v'_2, \dots, v'_b\}\big| + \big|E(H_2)\big| \geq \big|V(H_1 \cup H_2)\big|,
\end{aligned} \qquad (3.23)
$$

using $|E(H_2)| \geq |\{v'_1, v'_2, \dots, v'_b\}|$, since each distinct element in $\{v'_1, v'_2, \dots, v'_b\}$ contributes an edge to $E(H_2)$. This implies

$$
\big|\mathbb{E}\big[(T^+_{n,K,M})^a (T^{\pm}_{n,K,M})^b\big] - \mathbb{E}\big[(T^+_{n,K,M})^a Z^b_{n,K,M}\big]\big| \leq \sum_{\boldsymbol{e} \in \overline{\mathcal{M}}^{a,b}_{n,K}} p_n^{|V(H_1 \cup H_2)|},
$$

where $\overline{\mathcal{M}}^{a,b}_{n,K} \subseteq \mathcal{M}^{a,b}_{n,K}$ is the collection of all tuples in $\mathcal{M}^{a,b}_{n,K}$ such that

$$
\big|\{u_1, v_1, u_2, v_2, \dots, u_a, v_a, u'_1, u'_2, \dots, u'_b\}\big| + \big|E(H_2)\big| > \big|V(H_1 \cup H_2)\big|.
$$

Now, suppose that for every $s, t \in [b]$ such that $v'_s = v'_t$, we also have $u'_s = u'_t$. Then, $|E(H_2)| = |\{v'_1, \dots, v'_b\}|$, and hence, equality holds in (3.23). Therefore, $e \in \overline{\mathcal{M}}^{a,b}_{n,K}$ implies that there exist $s, t \in [b]$ such that $v'_s = v'_t$ and $u'_s \neq u'_t$, that is, the graph $H := H_1 \cup H_2$ must have a $K_{1,2}$ with central vertex in $V^-_{G_n,K}$. Recall that $V_M := \{v \in V(G_n) : d_v \leq M r_n\}$ and denote by $N_{a,b}(H, G_n[V_M])$ the number of ways of forming a graph isomorphic to H, with a edges from $G^+_{n,K}[V_M]$ and b edges from $G^{\pm}_{n,K}[V_M]$, such that the result graph contains a $K_{1,2}$, with central vertex in $V^-_{G_n,K}$ (and both edges in $E(G^{\pm}_{n,K}[V_M])$). Then

$$
\begin{aligned}
&\big|\mathbb{E}\big[(T^+_{n,K,M})^a (T^{\pm}_{n,K,M})^b\big] - \mathbb{E}\big[(T^+_{n,K,M})^a Z^b_{n,K,M}\big]\big| \\
&\quad \lesssim_{a,b} \sum_{H \in \mathcal{H}_{a,b}} N_{a,b}\big(H, G_n[V_M]\big) p_n^{|V(H)|},
\end{aligned} \qquad (3.24)
$$

where $\mathcal{H}_{a,b}$ is the set of all nonisomorphic graphs with at most $2(a+b)$ vertices, none of which is isolated, which contains at least one $K_{1,2}$ (the two-star) as a subgraph. Now, as in (3.20),

$$
N_{a,b}\big(H, G_n[V_M]\big) \lesssim_{a,b,M,H} r_n^{(|V(H)|-1)} d_{\lceil K r_n \rceil + 1}.
$$

This implies

$$
\lim_{K \to \infty} \lim_{n \to \infty} \frac{1}{r_n^{|V(H)|}} N_{a,b}\big(H, G_n[V_M]\big) = 0,
$$

by Observation 3.1. This completes the proof of the lemma, because the sum in the right hand side of (3.24) is over a finite set ($|\mathcal{H}_{a,b}| \leq 2^{\binom{2(a+b)}{2}}$).

Finally, by similar arguments as above and (3.24),

$$\mathbb{E}[(T^{+}_{n,K,M})^{a} Z^{b}_{n,K,M}] \leq \sum_{H \in \mathcal{H}_{a,b}} N_{a,b}(H, G_n[V_M]) p_n^{|V(H)|} \lesssim_{a,b,M} 1,$$

using the bound (3.18) and because the sum is over a finite set. □

Combining the above results, we get the following proposition which shows that $(T^{+}_{n,K,M}, T^{\pm}_{n,K,M}, T^{-}_{n,K,M})$ and $(T^{+}_{n,K,M}, Z_{n,K,M}, W^{-}_{n,K})$ are close in moments.

PROPOSITION 3.1. *Fix nonnegative integers a, b, c, and for $M > 0$. Then under the assumptions of Theorem 1.1,*

$$|\mathbb{E}[(T^{+}_{n,K,M})^{a} (T^{\pm}_{n,K,M})^{b} (T^{-}_{n,K,M})^{c}] - \mathbb{E}[(T^{+}_{n,K,M})^{a} Z^{b}_{n,K,M}]\mathbb{E}[(W^{-}_{n,K})^{c}]| \to 0,$$

as $n \to \infty$ followed by $K \to \infty$.

PROOF. Note that

$$\begin{aligned}
&|\mathbb{E}[(T^{+}_{n,K,M})^{a} (T^{\pm}_{n,K,M})^{b} (T^{-}_{n,K,M})^{c}] - \mathbb{E}[(T^{+}_{n,K,M})^{a} Z^{b}_{n,K,M}]\mathbb{E}[(W^{-}_{n,K})^{c}]| \\
&\quad \leq T_1 + T_2 + T_3,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &:= |\mathbb{E}[(T^{+}_{n,K,M})^{a} (T^{\pm}_{n,K,M})^{b} (T^{-}_{n,K,M})^{c}] - \mathbb{E}[(T^{+}_{n,K,M})^{a} (T^{\pm}_{n,K,M})^{b}]\mathbb{E}[(T^{-}_{n,K,M})^{c}]|, \\
T_2 &:= |\mathbb{E}[(T^{+}_{n,K,M})^{a} (T^{\pm}_{n,K,M})^{b}]\mathbb{E}[(T^{-}_{n,K,M})^{c}] - \mathbb{E}[(T^{+}_{n,K,M})^{a} Z^{b}_{n,K,M}]\mathbb{E}[(T^{-}_{n,K,M})^{c}]|, \\
T_3 &:= |\mathbb{E}[(T^{+}_{n,K,M})^{a} Z^{b}_{n,K,M}]\mathbb{E}[(T^{-}_{n,K,M})^{c}] - \mathbb{E}[(T^{+}_{n,K,M})^{a} Z^{b}_{n,K,M}]\mathbb{E}[(W^{-}_{n,K})^{c}]|.
\end{aligned}$$

Now, T_1 goes to zero (under the double limit) by Lemma 3.3, T_2 goes to zero by Lemma 3.4 (and using $\limsup_{K\to\infty} \limsup_{n\to\infty} \mathbb{E}[(T^{-}_{n,K,M})^{a}] \lesssim_a 1$ by Lemma 3.2), and T_3 goes to zero by Lemma 3.2 (and using $\limsup_{K\to\infty} \limsup_{n\to\infty} \mathbb{E}[(T^{+}_{n,K,M})^{a} Z^{b}_{n,K,M}] \lesssim_{a,b,M} 1$ by Lemma 3.4). □

3.4. *Convergence of moments and existence of limiting distribution.* Recall from (3.21)

$$Z_{n,K,M} = \sum_{u \in V^{+}_{G_n,K}} \sum_{v \in V^{-}_{G_n,K}} a_{uv}(G_n) J_{uv} X_{u,M} \overset{D}{=} \sum_{u \in V^{+}_{G_n,K}} J_u X_{u,M}, \tag{3.25}$$

where $J_u := \sum_{v \in V^{-}_{G_n,K}} a_{uv}(G_n) J_{uv} \sim \mathrm{Bin}(d^{\pm}_u, p_n)$ and $d^{\pm}_u := \sum_{v \in V^{-}_{G_n,K}} a_{uv}(G_n)$, is the number of edges between $u \in V^{+}_{G_n,K}$ and some vertex in $V^{-}_{G_n,K}$. Note that, by definition, $\{J_u\}_{u \in V^{+}_{G_n,K}}$ is a collection of independent Binomial random variables, independent of $\{X_v\}_{v \in V(G_n)}$. Next, define, $V^{+}_{n,K,M} := \{v \in V^{+}_{G_n,K} : d_v \leq M r_n\}$. Then (3.21) becomes (recall $X_{v,M} = X_v \mathbf{1}\{d_v \leq M r_n\}$ and $p_n = 1/r_n$),

$$Z_{n,K,M} = \sum_{u \in V^{+}_{n,K,M}} J_u X_u \sim \mathrm{Bin}\biggl(\sum_{u \in V^{+}_{n,K,M}} d^{\pm}_u X_u, \frac{1}{r_n}\biggr). \tag{3.26}$$

Define $Y_{n,K,M} = \frac{1}{r_n} \sum_{u \in V^{+}_{n,K,M}} d^{\pm}_u X_u$. The lemma below shows the existence of the limiting mixed moments of $(T^{+}_{n,K,M}, Y_{n,K,M})$. We begin with the following definition. Hereafter, we will assume that $K \geq 1$ is an integer. Also, denote by $\mathcal{W}_K$ the set of all symmetric measurable functions from $[0, K]^2 \to [0, 1]$. With these definitions, we now have the convergence of the mixed moments.

LEMMA 3.5. *Fix $K \geq 1$ and $M > 0$. Suppose there exist functions $W_K \in \mathcal{W}_K$, $d_K : [0, K] \mapsto [0, \infty)$, and measure-preserving bijections $\{\phi_{n,K}\}_{n=1}^{\infty}$ from $[0, K] \to [0, K]$, such that*

$$\lim_{n\to\infty} \| W_{G_n}^{\phi_{n,K}} - W_K \|_{\square([0,K]^2)} = 0 \tag{3.27}$$

and

$$\lim_{n\to\infty} \int_0^K \left| d_{W_{G_n}}^{\phi_{n,K}}(u) \mathbf{1}\{ d_{W_{G_n}}^{\phi_{n,K}}(u) \leq M \} - d_K(u) \mathbf{1}\{ d_K(u) \leq M \} \right| \mathrm{d}u = 0, \tag{3.28}$$

where

$$W_{G_n}^{\phi_{n,K}}(x, y) := W_{G_n}\big(\phi_{n,K}(x), \phi_{n,K}(y)\big) \mathbf{1}\{x, y \in [0, K]\}$$

and

$$d_{W_{G_n}}^{\phi_{n,K}}(x) := d_{W_{G_n}}\big(\phi_{n,K}(x)\big) \mathbf{1}\{x \in [0, K]\}.$$

Then $\mu_{a,b,K,M} := \lim_{n\to\infty} \mathbb{E}[(T_{n,K,M}^+)^a (Y_{n,K,M})^b]$ exists and is finite, for all nonnegative integers a, b.

In the proof of Theorem 1.1, this lemma will be used with $\phi_{n,K}$ as the identity map from $[0, K] \to [0, K]$, for all n. However, we will need the lemma in its generality for proving Theorem 1.2.

3.4.1. *Proof of Lemma* 3.5. We begin with the following definition:

DEFINITION 3.1. Given a graph $H = (V(H), E(H))$ (with possible isolated vertices), a function $W \in \mathcal{W}$, and $\boldsymbol{u} \in \mathbb{R}^{|V(H)|}$ define

$$t(H, W, \boldsymbol{u}) = \prod_{(a,b)\in E(H)} W(u_a, u_b),$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_{|V(H)|})$.

Now, recall from (3.3), $T_{n,K,M}^+ := \sum_{u,v \in V_{n,K,M}^+} a_{uv}(G_n) X_u X_v$. Let $\mathcal{N}_{n,K}^{a,b}$ be the collection of all $(a + b)$-tuples of the form

$$\boldsymbol{e} = \big((u_1, v_1), \ldots, (u_a, v_a), u_1', \ldots, u_b'\big),$$

where $u_1, v_1, \ldots, u_a, v_a, u_1', \ldots, u_b' \in V_{G_n,K}^+$, and $u_i \neq v_i$, for $1 \leq i \leq a$. Define the event,

$$\chi_{\boldsymbol{e}} = \mathbf{1}\{d_{u_i} \leq M r_n, d_{v_i} \leq M r_n, d_{u_j'} \leq M r_n, \text{for all } 1 \leq i \leq a, 1 \leq j \leq b\},$$

where d_v is the degree of the vertex labelled $v \in V(G_n)$. Then, recalling $Y_{n,K,M} = \frac{1}{r_n} \sum_{u \in V_{n,K,M}^+} d_u^{\pm} X_u$,

$$\begin{aligned} \mathbb{E}\big[(T_{n,K,M}^+)^a (Y_{n,K,M})^b\big] &= \sum_{\boldsymbol{e} \in \mathcal{N}_{n,K}^{a,b}} \mathbb{E}\left[\prod_{s=1}^a a_{u_s v_s}(G_n) X_{u_s} X_{v_s} \prod_{s=1}^b \frac{d_{u_s'}^{\pm}}{r_n} X_{u_s'} \right] \chi_{\boldsymbol{e}} \\ &= \sum_{\boldsymbol{e} \in \mathcal{N}_{n,K}^{a,b}} \frac{\chi_{\boldsymbol{e}}}{r_n^{|V(H)|}} \prod_{s=1}^a a_{u_s v_s}(G_n) \prod_{s=1}^b \frac{d_{u_s'}^{\pm}}{r_n}, \end{aligned} \tag{3.29}$$

where H is the graph formed by the union of the edges $(u_1, v_1), (u_2, v_2), \ldots, (u_a, v_a)$ and the vertices $u_1', u_2', \ldots, u_b'$.

Observe that

$$d^{\pm}_{u'_s} = d_{u'_s} - \sum_{j \in V^{+}_{G_n,K}} a_{u'_s j}(G_n).$$

Since $u'_s \in V^{+}_{G_n,K}$, there exists $x'_s \in [0, K]$ such that $u'_s = \lceil x'_s r_n \rceil$. This implies by (1.7) and (1.8),

$$\frac{1}{r_n} d^{\pm}_{u'_s} = d_{W_{G_n}}(x'_s) - \int_0^K W_{G_n}(x'_s, y)\,\mathrm{d}y + R_n(x'_s) = \zeta_{n,K}(x'_s) + R_n(x'_s), \tag{3.30}$$

where $\zeta_{n,K}(x) := (d_{W_{G_n}}(x) - \int_0^K W_{G_n}(x, y)\,\mathrm{d}y)$ and

$$R_n(x'_s) := -\int_K^{\frac{\lceil K r_n \rceil}{r_n}} W_{G_n}(x'_s, y)\,\mathrm{d}y. \tag{3.31}$$

Note that $\sup_x |R_n(x)| \le p_n$. Similarly, let $x_s, y_s \in [0, K]$ be such that $u_s = \lceil x_s r_n \rceil$ and $v_s = \lceil y_s r_n \rceil$. Then (recall (1.7))

$$\prod_{s=1}^{a} a_{u_s v_s}(G_n) = \prod_{s=1}^{a} a_{\lceil x_s r_n \rceil \lceil y_s r_n \rceil}(G_n) = \prod_{s=1}^{a} W_{G_n}(x_s, y_s). \tag{3.32}$$

Now, observe that the union of the edges $(u_1, v_1), \dots, (u_a, v_a)$ forms a graph $H_1 = (V(H_1), E(H_1))$, where $V(H_1) = \{u_1, \dots, u_a, v_1, \dots, v_a\}$ and

$$E(H_1) = \{(u_1, v_1), \dots, (u_a, v_a)\}.$$

Let $H = (V(H), E(H))$ be the graph obtained by the union of H_1 and the set of vertices $\{u'_1, \dots, u'_b\}$, that is, $V(H) = V(H_1) \cup \{u'_1, \dots, u'_b\}$ and $E(H) = E(H_1)$. Note that H has at most a edges and at most b isolated vertices. Let $\{w_1, w_2, \dots, w_{|V(H)|}\}$ be any labeling of the vertices in $V(H)$, and $\eta_j \in [0, b] := \{0, 1, 2, \dots, b\}$, for $1 \le j \le |V(H)|$, be the number of times the vertex w_j appears in the multi-set $\{u'_1, u'_2, \dots, u'_b\}$. Finally, let z_j be such that $w_j = \lceil z_j r_n \rceil$. Then using (3.30) and (3.32), for every graph H with at most a edges and at most b isolated vertices and every vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_{|V(H)|})$, there is a nonnegative constant $c(H, \boldsymbol{\eta})$, such that the sum in (3.29) can be rewritten as

$$\begin{aligned} &\mathbb{E}\big[(T^{+}_{n,K,M})^a (Y_{n,K,M})^b\big] \\ &\quad = \sum_{\substack{H \in \mathscr{G}_{a,b} \\ \boldsymbol{\eta} \in [0,b]^{|V(H)|}}} c(H, \boldsymbol{\eta}) \int_{\mathscr{B}_{K,n}} t(H, W_{G_n}, \boldsymbol{z}) \prod_{j=1}^{|V(H)|} \big(\zeta_{n,K}(z_j) + R_n(z_j)\big)^{\eta_j} \\ &\qquad \times \chi_{G_n,M}(z_j)\,\mathrm{d}z_j, \end{aligned} \tag{3.33}$$

where

- $\chi_{G_n,M}(z_j) := \mathbf{1}\{d_{W_{G_n}}(z_j) \le M\}$,
- $\mathscr{B}_{K,n} := [0, \frac{1}{r_n}\lceil K r_n \rceil]^{|V(H)|}$,
- $\boldsymbol{z} = (z_1, z_2, \dots, z_{|V(H)|})$,
- $\zeta_{n,K}(\cdot)$ is as in (3.30), $R_n(\cdot)$ as in (3.31), and $t(H, W_{G_n}, \boldsymbol{z})$ as in Definition 3.1,
- $\mathscr{G}_{a,b}$ is the collection of all graphs with at most a edges and at most b isolated vertices and $[0, b]^{|V(H)|} := \{0, 1, 2, \dots, b\}^{|V(H)|}$.

Note, since the sum in (3.33) is over a finite set (not depending on n) and each term in the integrand is bounded (by a function of H, K, and M), the integral over $\mathscr{B}_{K,n}$ can be replaced by the integral over $\mathscr{B}_K = [0, K]^{|V(H)|}$, as $n \to \infty$. Moreover, for every $1 \le j \le |V(H)|$, expanding the term $(\zeta_{n,K}(z_j) + R_n(z_j))^{\eta_j}$ in (3.33) by the binomial theorem, and using the fact $\sup_x |R_n(x)| \le p_n = o(1)$, the proof of Lemma 3.5 follows from Lemma 3.6 below.

LEMMA 3.6. *Under the assumptions of Lemma 3.5, given a finite simple graph $H = (V(H), E(H))$ (with possible isolated vertices), and the nonnegative integers $s_1, s_2, \dots, s_{|V(H)|}$,*

$$\lim_{n\to\infty} \int_{[0,K]^{|V(H)|}} t(H, W_{G_n}, \boldsymbol{u}) \prod_{a=1}^{|V(H)|} \zeta_{n,K}(u_a)^{s_a} \mathbf{1}\{d_{W_{G_n}}(u_a) \le M\}\, \mathrm{d}u_a$$
$$= \int_{[0,K]^{|V(H)|}} t(H, W_K, \boldsymbol{u}) \prod_{a=1}^{|V(H)|} \zeta_K(u_a)^{s_a} \mathbf{1}\{d_K(u_a) \le M\}\, \mathrm{d}u_a,$$

where $\zeta_{n,K}(x) := (d_{W_{G_n}}(x) - \int_0^K W_{G_n}(x,y)\,\mathrm{d}y)$ and $\zeta_K(x) := (d_K(x) - \int_0^K W_K(x,y)\,\mathrm{d}y)$.

PROOF. Define $d_{W_{G_n},K}(x) := \int_0^K W_{G_n}(x,y)\,\mathrm{d}y$ and $d_{W,K}(x) := \int_0^K W_K(x,y)\,\mathrm{d}y$. Expanding $\zeta_{n,K}(u_a)^{s_a} = (d_{W_{G_n}}(u_a) - d_{W_{G_n},K}(u_a))^{s_a}$ by the binomial theorem, for every $1 \le a \le |V(H)|$, we see it suffices to show that (recall $\mathscr{B}_K := [0,K]^{|V(H)|}$),

$$\lim_{n\to\infty} \int_{\mathscr{B}_K} t(H, W_{G_n}, \boldsymbol{u}) \prod_{a=1}^{|V(H)|} d_{W_{G_n}}(u_a)^{\kappa_a} d_{W_{G_n},K}(u_a)^{\lambda_a} \mathbf{1}\{d_{W_{G_n}}(u_a) \le M\}\, \mathrm{d}u_a$$
$$= \int_{\mathscr{B}_K} t(H, W_K, \boldsymbol{u}) \prod_{a=1}^{|V(H)|} d_K(u_a)^{\kappa_a} d_{W,K}(u_a)^{\lambda_a} \mathbf{1}\{d_K(u_a) \le M\}\, \mathrm{d}u_a, \tag{3.34}$$

for nonnegative integers $\kappa_1, \kappa_2, \dots, \kappa_{|V(H)|}$ and $\lambda_1, \lambda_2, \dots, \lambda_{|V(H)|}$.

To begin with, define

$$d^{\phi_{n,K}}_{W_{G_n},K}(x) := \int_0^K W_{G_n}\big(\phi_{n,K}(x), y\big)\,\mathrm{d}y = \int_0^K W_{G_n}\big(\phi_{n,K}(x), \phi_{n,K}(z)\big)\,\mathrm{d}z,$$

where the last equality follows by the change of variable $y = \phi_{n,K}(z)$. This implies,

$$\int_{\mathscr{B}_K} t(H, W_{G_n}, \boldsymbol{u}) \prod_{a=1}^{|V(H)|} d_{W_{G_n}}(u_a)^{\kappa_a} d_{W_{G_n},K}(u_a)^{\lambda_a} \mathbf{1}\{d_{W_{G_n}}(u_a) \le M\}\, \mathrm{d}u_a$$
$$= \int_{\mathscr{B}_K} t\big(H, W^{\phi_{n,K}}_{G_n}, \boldsymbol{z}\big) \prod_{a=1}^{|V(H)|} d^{\phi_{n,K}}_{W_{G_n}}(z_a)^{\kappa_a} d^{\phi_{n,K}}_{W_{G_n},K}(z_a)^{\lambda_a} \mathbf{1}\{d^{\phi_{n,K}}_{W_{G_n}}(z_a) \le M\}\, \mathrm{d}z_a, \tag{3.35}$$

by changes of variables $u_a = \phi_{n,K}(z_a)$, for $1 \le a \le |V(H)|$, where $\boldsymbol{u} = (u_1, u_2, \dots, u_{|V(H)|})$ and $\boldsymbol{z} = (z_1, z_2, \dots, z_{|V(H)|})$. Therefore, by (3.34), it suffices to show that

$$\lim_{n\to\infty} \int_{\mathscr{B}_K} t\big(H, W^{\phi_{n,K}}_{G_n}, \boldsymbol{z}\big) \prod_{a=1}^{|V(H)|} d^{\phi_{n,K}}_{W_{G_n}}(z_a)^{\kappa_a} d^{\phi_{n,K}}_{W_{G_n},K}(z_a)^{\lambda_a} \mathbf{1}\{d^{\phi_{n,K}}_{W_{G_n}}(z_a) \le M\}\, \mathrm{d}z_a$$
$$= \int_{\mathscr{B}_K} t(H, W_K, \boldsymbol{u}) \prod_{a=1}^{|V(H)|} d_K(u_a)^{\kappa_a} d_{W,K}(u_a)^{\lambda_a} \mathbf{1}\{d_K(u_a) \le M\}\, \mathrm{d}u_a. \tag{3.36}$$

Now, denote $d^{\phi_{n,K}}_{W_{G_n}}(u_a|M) := d^{\phi_{n,K}}_{W_{G_n}}(u_a)\mathbf{1}\{d^{\phi_{n,K}}_{W_{G_n}}(u_a) \le M\}$. Then by a telescoping argument similar to the proof of [8], Theorem 3.7(a), it follows that

$$\left| \int_{\mathscr{B}_K} \big(t\big(H, W^{\phi_{n,K}}_{G_n}, \boldsymbol{u}\big) - t(H, W_K, \boldsymbol{u})\big) \prod_{a=1}^{|V(H)|} d^{\phi_{n,K}}_{W_{G_n},K}(u_a)^{\lambda_a} d^{\phi_{n,K}}_{W_{G_n}}(u_a|M)^{\kappa_a}\, \mathrm{d}u_a \right|$$
$$\lesssim_{M,H,K} \big\| W^{\phi_{n,K}}_{G_n} - W_K \big\|_{\square([0,K]^2)}. \tag{3.37}$$

Moreover, recalling $d^{\phi_{n,K}}_{W_{G_n},K}(x) := \int_0^K W_{G_n}(\phi_{n,K}(x), y)\,\mathrm{d}y = \int_0^K W_{G_n}(\phi_{n,K}(x), \phi_{n,K}(z))\,\mathrm{d}z$, and from the definition of the cut-distance,

$$\sup_{f:[0,K]\to[-1,1]}\left|\int_0^K \big(d^{\phi_{n,K}}_{W_{G_n},K}(u) - d_{W,K}(u)\big) f(u)\,\mathrm{d}u\right| \le \big\|W^{\phi_{n,K}}_{G_n} - W_K\big\|_{\square([0,K]^2)}.$$

Then, for any integer $a \ge 1$ and all $f : [0, K] \to [-1, 1]$,

$$\begin{aligned} &\left|\int_0^K \big(d^{\phi_{n,K}}_{W_{G_n},K}(u)^a - d_{W,K}(u)^a\big) f(u)\,\mathrm{d}u\right| \\ &\quad\le \left|\int_0^K \big(d^{\phi_{n,K}}_{W_{G_n},K}(u) - d_{W,K}(u)\big) d^{\phi_{n,K}}_{W_{G_n},K}(u)^{a-1} f(u)\,\mathrm{d}u\right| \\ &\qquad + \left|\int_0^K \big(d^{\phi_{n,K}}_{W_{G_n},K}(u)^{a-1} - d_{W,K}(u)^{a-1}\big) d_{W,K}(u) f(u)\,\mathrm{d}u\right| \\ &\quad\lesssim_{M,H,K} \big\|W^{\phi_{n,K}}_{G_n} - W_K\big\|_{\square([0,K]^2)}, \end{aligned} \tag{3.38}$$

where the last step follows by repeating the telescoping argument $a - 1$ times. Now, define

$$A_1(\boldsymbol{u}) := t(H, W_K, \boldsymbol{u}) \prod_{a=1}^{|V(H)|} d^{\phi_{n,K}}_{W_{G_n},K}(u_a)^{\lambda_a} \prod_{a=1}^{|V(H)|} d^{\phi_{n,K}}_{W_{G_n}}(u_a|M)^{\kappa_a}$$

and

$$A_2(\boldsymbol{u}) := t(H, W_K, \boldsymbol{u}) \prod_{a=1}^{|V(H)|} d_{W,K}(u_a)^{\lambda_a} \prod_{a=1}^{|V(H)|} d^{\phi_{n,K}}_{W_{G_n}}(u_a|M)^{\kappa_a}.$$

Then repeating the telescoping argument again gives,

$$\left|\int_{\mathscr{B}_K} \big(A_1(\boldsymbol{u}) - A_2(\boldsymbol{u})\big) \prod_{a=1}^{|V(H)|} \mathrm{d}u_a\right| \lesssim_{M,H,K} \big\|W^{\phi_{n,K}}_{G_n} - W_K\big\|_{\square([0,K]^2)}. \tag{3.39}$$

Hence, combining (3.37) and (3.39), and the triangle inequality gives,

$$\begin{aligned} &\left|\int_{\mathscr{B}_K} \left(t\big(H, W^{\phi_{n,K}}_{G_n}, \boldsymbol{u}\big) \prod_{a=1}^{|V(H)|} d^{\phi_{n,K}}_{W_{G_n},K}(u_a)^{\lambda_a} d^{\phi_{n,K}}_{W_{G_n}}(u_a|M)^{\kappa_a} - A_2(\boldsymbol{u})\right) \prod_{a=1}^{|V(H)|} \mathrm{d}u_a\right| \\ &\quad\lesssim_{M,H,K} \big\|W^{\phi_{n,K}}_{G_n} - W_K\big\|_{\square([0,K]^2)}. \end{aligned} \tag{3.40}$$

Note that the RHS above goes to zero as $n \to \infty$, by (3.27).

Next, define $d_K(u_a|M) := d_K(u_a)\mathbf{1}\{d_K(u_a) \le M\}$, and note that

$$\begin{aligned} &\left|\int_{\mathscr{B}_K} \left(A_2(\boldsymbol{u}) - t(H, W_K, \boldsymbol{u}) \prod_{a=1}^{|V(H)|} d_{W,K}(u_a)^{\lambda_a} d_K(u_a|M)^{\kappa_a}\right) \prod_{a=1}^{|V(H)|} \mathrm{d}u_a\right| \\ &\quad\lesssim_{M,H,K} \int_{\mathscr{B}_K} \left|\prod_{a=1}^{|V(H)|} d^{\phi_{n,K}}_{W_{G_n}}(u_a|M)^{\kappa_a} - \prod_{a=1}^{|V(H)|} d_K(u_a|M)^{\kappa_a}\right| \prod_{a=1}^{|V(H)|} \mathrm{d}u_a \\ &\quad\lesssim_{M,H,K} \int_0^K \big|d^{\phi_{n,K}}_{W_{G_n}}(u|M) - d_K(u|M)\big|\,\mathrm{d}u, \end{aligned} \tag{3.41}$$

where the last step follows by a telescoping argument (similar to Observation 3.2 below). Note that RHS above goes to zero as $n \to \infty$, by (3.28).

Therefore, taking the limit as $n \to \infty$, and combining (3.40) and (3.41), and the triangle inequality, gives (3.36), as required. $\square$

The next observation shows that a sufficient condition for (3.28) to hold infinitely often is the L_1 convergence of the function $d_{W_{G_n}}^{\phi_n,K}$ to d_K. To this end, we need a definition.

DEFINITION 3.2. Let $\mathcal{D}$ denote the set of all positive reals M such that for all positive integers K, $\mathbb{P}(d_K(U_K) = M) = 0$, where $U_K \sim \mathrm{Unif}[0, K]$.

Note that the complement of $\mathcal{D}$ in $(0, \infty)$ is countable, and so given any $M_0 > 0$ we can choose $M > M_0$ with $M \in \mathcal{D}$.

OBSERVATION 3.2. *Suppose*

$$\lim_{n\to\infty} \int_0^K \left|d_{W_{G_n}}^{\phi_n,K}(x) - d_K(x)\right| \mathrm{d}x = 0. \tag{3.42}$$

Then any integer $a \geq 1$,

$$\lim_{n\to 0} \int_0^K \left|d_{W_{G_n}}^{\phi_n,K}(u)^a \mathbf{1}\{d_{W_{G_n}}^{\phi_n,K}(u) \leq M\} - d_K(u)^a \mathbf{1}\{d_K(u) \leq M\}\right| \mathrm{d}u = 0, \tag{3.43}$$

whenever $M \in \mathcal{D}$.

PROOF. To begin with suppose $a = 1$. The assumption (3.42) implies, $d_{W_{G_n}}^{\phi_n,K}(U) \stackrel{L_1}{\to} d_K(U)$, where $U \sim \mathrm{Unif}[0, K]$. Note that $\mathbb{E}(d_K(U)) < \infty$. (To observe this, note that, for all $K \geq 1$, $\int_0^K d_K(x)\,\mathrm{d}x = \limsup_{n\to\infty} \int_0^K d_{W_{G_n}}^{\phi_n,K}(x)\,\mathrm{d}x \lesssim \limsup_{n\to\infty} \frac{1}{r_n^2}|E(G_n)|$, which is bounded by (1.6).) Then for every sequence there is a further subsequence $\{n_s\}_{s\geq 1}$ along which

$$d_{W_{G_{n_s}}}^{\phi_{n_s},K}(U) \stackrel{\text{a.s.}}{\to} d_K(U).$$

Hence, along this subsequence, $d_{W_{G_{n_s}}}^{\phi_{n_s},K}(U)\mathbf{1}\{d_{W_{G_{n_s}}}^{\phi_{n_s},K}(U) \leq M\} \stackrel{\text{a.s.}}{\to} d_K(U)\mathbf{1}\{d_K(U) \leq M\}$, whenever $\mathbb{P}(d_K(U) = M) = 0$, that is, $M \in \mathcal{D}$. Then by the dominated convergence theorem,

$$d_{W_{G_{n_s}}}^{\phi_{n_s},K}(U)\mathbf{1}\{d_{W_{G_{n_s}}}^{\phi_{n_s},K}(U) \leq M\} \stackrel{L_1}{\to} d_K(U)\mathbf{1}\{d_K(U) \leq M\},$$

proving (3.43) for $a = 1$.

Define, $d_K(u|M) := d_K(u)\mathbf{1}\{d_K(u) \leq M\}$. Then for $a > 1$, a telescoping argument gives,

$$\begin{aligned}
&\int_0^K \left|d_{W_{G_n}}^{\phi_n,K}(u)^a \mathbf{1}\{d_{W_{G_n}}^{\phi_n,K}(u) \leq M\} - d_K(u)^a \mathbf{1}\{d_K(u) \leq M\}\right| \mathrm{d}u \\
&\quad \leq \int_0^K d_{W_{G_n}}^{\phi_n,K}(u)^{a-1} \mathbf{1}\{d_{W_{G_n}}^{\phi_n,K}(u) \leq M\} \left|d_{W_{G_n}}^{\phi_n,K}(u)\mathbf{1}\{d_{W_{G_n}}^{\phi_n,K}(u) \leq M\} - d_K(u|M)\right| \mathrm{d}u \\
&\quad\quad + \int_0^K \left|d_{W_{G_n}}^{\phi_n,K}(u)^{a-1} d_K(u)\mathbf{1}\{d_{W_{G_n}}^{\phi_n,K}(u), d_K(u) \leq M\} - d_K(u)^a \mathbf{1}\{d_K(u) \leq M\}\right| \mathrm{d}u \\
&\quad \lesssim_{M,a} \int_0^K \left|d_{W_{G_n}}^{\phi_n,K}(u)\mathbf{1}\{d_{W_{G_n}}^{\phi_n,K}(u) \leq M\} - d_K(u|M)\right| \mathrm{d}u \to 0,
\end{aligned}$$

where the second inequality follows by repeating the telescoping argument from the previous step $a - 1$ times, and the last step uses (3.43) for $a = 1$. $\square$

3.4.2. *Existence of limit of* $(T^+_{n,K,M}, Y_{n,K,M})$. The existence of the limiting distribution of $(T^+_{n,K,M}, Y_{n,K,M})$ follows from the above lemma and the Carleman moment condition.

LEMMA 3.7. *Suppose the assumptions of Lemma 3.5 hold. Then there exists random variables* $(T^+_{K,M}, Y_{K,M})$ *such that, as* $n \to \infty$,

$$(T^+_{n,K,M}, Y_{n,K,M}) \to (T^+_{K,M}, Y_{K,M}),$$

in distribution and in all (*mixed*) *moments*.

PROOF. Recall that $V^+_{n,K,M} := \{v \in V^+_{G_n,K} : d_v \le M r_n\}$. Then from (3.26),

$$Y_{n,K,M} = \frac{1}{r_n} \sum_{u \in V^+_{n,K,M}} d^{\pm}_u X_u \le M \sum_{u \in V^+_{n,K,M}} X_u \sim M \operatorname{Bin}(|V^+_{n,K,M}|, p_n).$$

Note that $|V^+_{n,K,M}| \le \lceil K/p_n \rceil$, which implies that $Y_{n,K,M}$ is stochastically dominated by the random variable $M \operatorname{Bin}(\lceil K/p_n \rceil, p_n)$. This implies, since $\mu_{0,b,K,M} = \lim_{n\to\infty} \mathbb{E}[Y^b_{n,K,M}]$ exists (by Lemma 3.5), for all $b \ge 1$,

$$\mu_{0,b,K,M} = \lim_{n\to\infty} \mathbb{E}[Y^b_{n,K,M}] \le (MCKb)^b, \tag{3.44}$$

using the bound $\mathbb{E}[\operatorname{Bin}(n,p)^a] \le C^a (\frac{a}{\log a})^a \max\{np, (np)^a\} \le C^a a^a \max\{np, (np)^a\}$, for $a \ge 3$ and some universal constant $C < \infty$, [22], Corollary 3.

Next, define $S_{n,K,M} = \frac{1}{2} \sum_{u \in V^+_{n,K,M}} \sum_{v \in V^+_{n,K,M} \setminus \{u\}} X_u X_v$. Then,

$$S_{n,K,M} \stackrel{D}{=} \binom{R_{n,K,M}}{2} \quad \text{where } R_{n,K,M} \sim \operatorname{Bin}(|V^+_{n,K,M}|, p_n),$$

and $T^+_{n,K,M} \le S_{n,K,M}$. Again using $|V^+_{n,K,M}| \le \lceil K r_n \rceil$ and Lemma 3.5, it follows that, for all $a \ge 3$,

$$\mu_{a,0,K,M} = \lim_{n\to\infty} \mathbb{E}[(T^+_{n,K,M})^a] \le \limsup_{n\to\infty} \mathbb{E}[R^{2a}_{n,K,M}] \le (2CKa)^{2a}, \tag{3.45}$$

using bounds on moments of the binomial distribution [22], Corollary 3, as in (3.44).

Combining (3.44) and (3.45) gives,

$$\sum_{a=1}^{\infty} \frac{1}{(\mu_{a,0,K,M} + \mu_{0,a,K,M})^{\frac{1}{2a}}} \ge \frac{1}{\sqrt{2}} \sum_{a=1}^{\infty} \frac{1}{\max\{2CKa, \sqrt{MCKa}\}} = \infty.$$

Therefore, by the Carleman condition for multivariate distributions [19, 28] (recall that the existence of the limiting mixed moments $\mu_{a,b,K,M}$, for all positive integers a, b, follows from Lemma 3.5), implies that $(T^+_{n,K,M}, Y_{n,K,M})$ converges in distribution and in all mixed moments to some random variable $(T^+_{K,M}, Y_{K,M})$. This completes the proof. □

3.5. *Deriving the limiting distribution of* $(T^+_{n,K,M}, Z_{n,K,M})$. Let G_n be a sequence of graphs satisfying the assumptions of Lemma 3.5, with the functions $W_K : [0,K]^2 \to [0,1]$ and $d_K : [0,K] \to [0,1]$. Denote by

$$W_{K,M}(x,y) = W_K(x,y) \mathbf{1}\{\max\{d_K(x), d_K(y)\} \le M\}. \tag{3.46}$$

For $W_{K,M}$, define its L-step piecewise constant approximation (note that it has K^2L^2 blocks) as follows:

$$W^{(L)}_{K,M}(x,y) := \sum_{1 \le a,b \le KL} r^{(L)}_{K,M}(a,b) \mathbf{1}\left\{x \in \left(\frac{a-1}{L}, \frac{a}{L}\right]\right\} \mathbf{1}\left\{y \in \left(\frac{b-1}{L}, \frac{b}{L}\right]\right\},$$

where

$$r_{K,M}^{(L)}(a,b) := L^2 \int_{\frac{a-1}{L}}^{\frac{a}{L}} \int_{\frac{b-1}{L}}^{\frac{b}{L}} W_{K,M}(u,v)\,\mathrm{d}u\,\mathrm{d}v. \tag{3.47}$$

By Proposition A.1,

$$\lim_{L\to\infty} \|W_{K,M}^{(L)} - W_{K,M}\|_{\square([0,K]^2)} \leq \lim_{L\to\infty} \|W_{K,M}^{(L)} - W_{K,M}\|_{L_1([0,K]^2)} \to 0.$$

DEFINITION 3.3. Given a function $H : [0,K]^2 \to [0,1]$ and a positive integer N, the H-random graph on N vertices (denoted by $\mathcal{G}(N,H)$) is the simple undirected labelled random graph with vertex set $[N] := \{1,2,\ldots,N\}$ and edges are present independently, with

$$\mathbb{P}((u,v) \in E(\mathcal{G}(N,H)) = H\left(\frac{Ku}{N}, \frac{Kv}{N}\right) \quad \text{for } 1 \leq u < v \leq N.$$

Fix $K \geq 1$. Let $G_{n,K,M}^{(L)}$ be the $W_{K,M}^{(L)}$-random graph $\mathcal{G}(\lceil Kr_n \rceil, W_{K,M}^{(L)})$, independent of $\{X_v\}_{v\in V(G_n)}$. For $u \in [0,K]$, define the function

$$\Delta_{K,M}(u) := \left(d_K(u) - \int_0^K W_K(u,v)\,\mathrm{d}v\right)\mathbf{1}\{d_K(u) \leq M\}. \tag{3.48}$$

Define the L-step approximation of $\Delta_{K,M}^{(L)}$ as follows:

$$\Delta_{K,M}^{(L)}(x) = \sum_{a=1}^{KL} \eta_{K,M}^{(L)}(a)\mathbf{1}\left\{x \in \left(\frac{a-1}{L}, \frac{a}{L}\right]\right\} = \eta_{K,M}^{(L)}(\lceil Lx \rceil), \tag{3.49}$$

where $\eta_{K,M}^{(L)}(a) := L\int_{\frac{(a-1)}{L}}^{\frac{a}{L}} \Delta_{K,M}(u)\,\mathrm{d}u$. By Proposition A.1, $\|\Delta_{K,M}^{(L)} - \Delta_{K,M}\|_{L_1([0,K])} \to 0$, as $L \to \infty$.

Recall that $A(G_{n,K,M}^{(L)}) = ((A(G_{n,K,M}^{(L)})(u,v)))_{1\leq u,v\leq \lceil Kr_n \rceil}$ is the adjacency matrix of the graph $G_{n,K,M}^{(L)}$. Let $N = \lceil Kr_n \rceil$ and define

$$\begin{aligned} \overline{T}_{n,L,K,M}^{+} &:= \sum_{1\leq u<v\leq N} A(G_{n,K,M}^{(L)})(u,v)X_u X_v, \\ \overline{Y}_{n,L,K,M} &:= \sum_{u=1}^{N} \eta_{K,M}^{(L)}\left(\left\lceil \frac{KLu}{N} \right\rceil\right) X_u. \end{aligned} \tag{3.50}$$

LEMMA 3.8. *Fix an integer $K \geq 1$ and $M > 0$. Under the assumptions of Lemma 3.5, for $t_1, t_2 \geq 0$,*

$$\begin{aligned} &\lim_{L\to\infty}\lim_{n\to\infty} \mathbb{E}\exp\{-t_1\overline{T}_{n,L,K,M}^{+} - t_2\overline{Y}_{n,L,K,M}\} \\ &\quad = \mathbb{E}\exp\left\{\frac{1}{2}\int_0^K\int_0^K \phi_{t_1,K,M}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y) - t_2\int_0^K \Delta_{K,M}(x)\,\mathrm{d}N(x)\right\}, \end{aligned} \tag{3.51}$$

where

- $\{N(t), t \geq 0\}$ *is a homogenous Poisson process of rate* 1,
- $\phi_{t_1,K,M}(x,y) := \log(1 - W_{K,M}(x,y) + W_{K,M}(x,y)e^{-t_1})$, *where $W_{K,M}$ is as defined in* (3.46), *and*
- $\Delta_{K,M}(x)$ *as in* (3.48).

PROOF. Throughout the proof denote $N = \lceil Kr_n \rceil$. The linear part (recall (3.50)) can be written as

$$\overline{Y}_{n,L,K,M} = \sum_{a=1}^{KL} \eta_{K,M}^{(L)}(a) X_n(a), \tag{3.52}$$

where $X_n(a) := \sum_{u=1}^{N} \mathbf{1}\{\lceil \frac{KLu}{N} \rceil = a\} X_u$. Note that

$$X_n(a) \sim \text{Bin}\left(\sum_{u=1}^{N} \mathbf{1}\left\{\left\lceil \frac{KLu}{N} \right\rceil = a\right\}, p_n\right), \tag{3.53}$$

and $\{X_n(a)\}_{1 \le a \le KL}$ are mutually independent.

For notational brevity, take $\sigma_n = K/N = K/\lceil Kr_n \rceil$. For the quadratic term, taking an expectation over the random graph $G_{n,K,M}^{(L)}$ we get,

$$\begin{aligned}
&\mathbb{E}\big[e^{-t_1 \overline{T}^{+}_{n,L,K,M}} | X_1, \dots, X_N\big] \\
&\quad = \prod_{1 \le u < v \le N} \mathbb{E}\big[e^{-t_1 A(G_{n,K,M}^{(L)})(u,v) X_u X_v} | X_1, \dots, X_N\big] \\
&\quad = \prod_{1 \le u < v \le N} \big(1 - W_{K,M}^{(L)}(\sigma_n u, \sigma_n v) + W_{K,M}^{(L)}(\sigma_n u, \sigma_n v) e^{-t_1 X_u X_v}\big) \\
&\quad = \prod_{1 \le u < v \le N} \big(1 - W_{K,M}^{(L)}(\sigma_n u, \sigma_n v) + W_{K,M}^{(L)}(\sigma_n u, \sigma_n v) e^{-t_1}\big)^{X_u X_v},
\end{aligned}$$

where the last equality uses the fact that $X_u X_v$ is a Bernoulli random variable. Using the definition of $W_{K,M}^{(L)}$, the RHS above equals

$$\prod_{1 \le a < b \le KL} \big(\varphi_{t_1,L,K}(a,b)\big)^{X_n(a) X_n(b)} \prod_{a=1}^{KL} \big(\varphi_{t_1,L,K}(a,a)\big)^{\binom{X_n(a)}{2}}, \tag{3.54}$$

where $\varphi_{t_1,L,K}(a,b) := 1 - r_{K,M}^{(L)}(a,b) + r_{K,M}^{(L)}(a,b) e^{-t_1}$ (recall (3.47)).

On letting $n \to \infty$, we have

$$\big\{X_n(1), X_n(2), \dots, X_n(KL)\big\} \xrightarrow{D} \big\{\partial N(1), \partial N(2), \dots, \partial N(KL)\big\},$$

where $\{N(t) : 0 \le t \le K\}$ is a Poisson process of rate 1 and $\partial N(a) := N(\frac{a}{L}) - N(\frac{a-1}{L}) \sim \text{Pois}(1/L)$ (by (3.53) and the Poisson approximation to the binomial distribution). Note that $\{\partial N(1), \partial N(2), \dots, \partial N(KL)\}$ is independent, since increments of the Poisson process are independent. Therefore, by (3.52), (3.54) and the continuous mapping theorem, as $n \to \infty$,

$$\big(\overline{Y}_{n,L,K,M}, \mathbb{E}\big[e^{-t_1 \overline{T}^{+}_{n,L,K,M}} | X_1, \dots, X_N\big]\big) \xrightarrow{D} (\psi_{L,K,M}, \theta_{L,K,M}), \tag{3.55}$$

where $\psi_{L,K,M} := \sum_{a=1}^{KL} \eta_{K,M}^{(L)}(a) \partial N(a)$ and

$$\theta_{L,K,M} := \prod_{1 \le a < b \le KL} \big(\varphi_{t_1,L,K}(a,b)\big)^{\partial N(a) \partial N(b)} \prod_{a=1}^{KL} \big(\varphi_{t_1,L,K}(a,a)\big)^{\binom{\partial N(a)}{2}}.$$

For $x, y \in [0, K]$, defining

$$\begin{aligned}
\phi_{t_1,L,K,M}(x,y) &:= \log \varphi_{t_1,L,K}\big(\lceil Lx \rceil, \lceil Ly \rceil\big) \quad \text{if } \lceil Lx \rceil \ne \lceil Ly \rceil, \\
&:= 0 \quad \text{if } \lceil Lx \rceil = \lceil Ly \rceil
\end{aligned}$$

gives,

$$\log \theta_{L,K,M} = \frac{1}{2} \sum_{1 \le a \ne b \le KL} \partial N(a) \partial N(b) \log \varphi_{t_1,L,K}(a,b) + \sum_{a=1}^{KL} \binom{\partial N(a)}{2} \log \varphi_{t_1,L,K}(a,a)$$

$$= \frac{1}{2} \int_{[0,K]^2} \phi_{t_1,L,K,M}(x,y)\, \mathrm{d}N(x)\, \mathrm{d}N(y) + \sum_{a=1}^{KL} \binom{\partial N(a)}{2} \log \varphi_{t_1,L,K}(a,a), \tag{3.56}$$

using the definition of the stochastic integral for elementary functions (Definition B.1).

To begin with we consider the first term in (3.56) above. Recall the definition of $\phi_{t_1,K,M}(x,y)$ from the statement of the Lemma 3.8. Using

$$\lim_{L\to\infty} \| W_{K,M}^{(L)} - W_{K,M} \|_{L_1([0,K]^2)} = 0, \tag{3.57}$$

and the dominated convergence theorem (note that the functions $\phi_{t_1,L,K,M}$ and $\phi_{t_1,K,M}$ are bounded above by $\log(1 + e^{-t_1})$ and bounded below by $-t_1$), gives $\|\phi_{t_1,L,K,M}(x,y) - \phi_{t_1,K,M}(x,y)\|_{L_1([0,K]^2)} \to 0$, as $L \to \infty$, for every $t_1 \ge 0$ fixed. Then, by Proposition B.2, as $L \to \infty$,

$$\frac{1}{2} \int_{[0,K]^2} \phi_{t_1,L,K,M}(x,y)\, \mathrm{d}N(x)\, \mathrm{d}N(y) \xrightarrow{P} \frac{1}{2} \int_{[0,K]^2} \phi_{t_1,K,M}(x,y)\, \mathrm{d}N(x)\, \mathrm{d}N(y). \tag{3.58}$$

Next, consider the second term in (3.56): Using $\mathbb{E}\binom{\partial N(a)}{2} \lesssim 1/L^2$ and $\sup_a |\log \varphi_{t_1,L,K}(a, a)| \lesssim_{t_1} 1$ gives,

$$\sum_{a=1}^{KL} \mathbb{E} \binom{\partial N(a)}{2} \log \varphi_{t_1,L,K}(a,a) \lesssim_{K,t_1} \frac{1}{L}.$$

Therefore,

$$\sum_{a=1}^{KL} \binom{\partial N(a)}{2} \log \varphi_{t_1,L,K}(a,a) \xrightarrow{L_1} 0, \tag{3.59}$$

as $L \to \infty$. The limits in (3.58) and (3.59) combined with (3.56) gives,

$$\log \theta_{L,K,M} \xrightarrow{P} \frac{1}{2} \int_{[0,K]^2} \phi_{t_1,K,M}(x,y)\, \mathrm{d}N(x)\, \mathrm{d}N(y). \tag{3.60}$$

Similarly, as $L \to \infty$,

$$\psi_{L,K,M} := \sum_{a=1}^{KL} \eta_{K,M}^{(L)}(a) \partial N(a) \xrightarrow{P} \int_0^K \Delta_{K,M}(x)\, \mathrm{d}N(x). \tag{3.61}$$

Combining (3.60) and (3.61) with (3.55), and another application of the dominated convergence theorem completes the proof of the lemma. □

Next, we show that the limiting distribution of $(T_{n,K,M}^+, Y_{n,K,M})$ is same as that of $(\overline{T}_{n,L,K,M}^+, \overline{Y}_{n,L,K,M})$ derived above.

LEMMA 3.9. *Fix $K, M \ge 1$. Under the assumptions of Lemma 3.5, $(T_{n,K,M}^+, Y_{n,K,M})$ converge in distribution and in moments, as $n \to \infty$, to $(T_{K,M}^+, Y_{K,M})$, with joint moment generating function*

$$\mathbb{E} \exp\{-t_1 T_{K,M}^+ - t_2 Y_{K,M}\}$$

$$= \mathbb{E} \exp\left\{\frac{1}{2} \int_0^K \int_0^K \phi_{t_1,K,M}(x,y)\, \mathrm{d}N(x)\, \mathrm{d}N(y) - t_2 \int_0^K \Delta_{K,M}(x)\, \mathrm{d}N(x)\right\}, \tag{3.62}$$

with $t_1, t_2 \geq 0$, $\Delta_{K,M}(\cdot)$, $\{N(t), t \geq 0\}$, *and* $\phi_{t_1,K,M}(\cdot,\cdot)$ *are as defined in Lemma* 3.8. *Moreover*, $(T^+_{n,K,M}, Z_{n,K,M})$ *converge in distribution and in moments, as* $n \to \infty$, *to* $(T^+_{K,M}, Z_{K,M})$, *with joint moment generating function*

$$\begin{aligned}
&\mathbb{E}\exp\{-t_1 T^+_{K,M} - t_2 Z_{K,M}\} \\
&\quad = \mathbb{E}\exp\left\{\frac{1}{2}\int_0^K \int_0^K \phi_{t_1,K,M}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y) - \hat{t}_2 \int_0^K \Delta_{K,M}(x)\,\mathrm{d}N(x)\right\},
\end{aligned} \tag{3.63}$$

where $\hat{t}_2 = 1 - e^{-t_2}$.

PROOF. We begin by computing $\lim_{L\to\infty}\lim_{n\to\infty}\mathbb{E}[(\overline{T}^+_{n,L,K,M})^a(\overline{Y}_{n,L,K,M})^b]$. To this end, let $\mathcal{N}^{a,b}_N$ be the collection of all $(a+b)$-tuples of the form

$$\boldsymbol{e} = \big((u_1, v_1), \dots, (u_a, v_a), u'_1, \dots, u'_b\big),$$

where $u_1, v_1, \dots, u_a, v_a, u'_1, \dots, u'_b \in [N] := \{1, 2, \dots, N\}$, and $u_i < v_i$, for $1 \leq i \leq a$. Then recalling (3.50), it follows that

$$\begin{aligned}
&\mathbb{E}\big[\big(\overline{T}^+_{n,L,K,M}\big)^a\big(\overline{Y}_{n,L,K,M}\big)^b\big] \\
&\quad = \sum_{\boldsymbol{e}\in\mathcal{N}^{a,b}_N} \mathbb{E}\left[\prod_{s=1}^a A\big(G^{(L)}_{n,K,M}\big)(u_s, v_s) X_{u_s} X_{v_s} \prod_{s=1}^b \eta^{(L)}_{K,M}\left(\left\lceil \frac{KLu'_s}{N}\right\rceil\right) X_{u'_s}\right] \\
&\quad = \sum_{\boldsymbol{e}\in\mathcal{N}^{a,b}_N} \frac{1}{r_n^{|V(H)|}} \mathbb{E}\left[\prod_{s=1}^a A\big(G^{(L)}_{n,K,M}\big)(u_s, v_s)\right] \prod_{s=1}^b \eta^{(L)}_{K,M}\left(\left\lceil \frac{KLu'_s}{N}\right\rceil\right) \\
&\quad = \sum_{\boldsymbol{e}\in\mathcal{N}^{a,b}_N} \frac{1}{r_n^{|V(H)|}} \prod_{s=1}^a W^{(L)}_{K,M}\left(\frac{Ku_s}{N}, \frac{Kv_s}{N}\right) \prod_{s=1}^b \eta^{(L)}_{K,M}\left(\left\lceil \frac{KLu'_s}{N}\right\rceil\right),
\end{aligned} \tag{3.64}$$

where H is the graph formed by the union of the edges $(u_1, v_1), (u_2, v_2), \dots, (u_a, v_a)$ and the vertices $u'_1, u'_2, \dots, u'_b$. Note that since $u'_s \in [N] = [\lceil Kr_n\rceil]$, there exists $x'_s \in [0, K]$ such that $u'_s = \lceil x'_s r_n\rceil$. This implies

$$\eta^{(L)}_{K,M}\left(\left\lceil \frac{KLu'_s}{N}\right\rceil\right) = \eta^{(L)}_{K,M}\left(\left\lceil \frac{KL\lceil x'_s r_n\rceil}{\lceil Kr_n\rceil}\right\rceil\right) := \hat{\Delta}^{(L)}_{n,K,M}(x'_s). \tag{3.65}$$

Similarly, let $x_s, y_s \in [0, K]$ be such that $u_s = \lceil x_s r_n\rceil$ and $v_s = \lceil y_s r_n\rceil$. Then

$$W^{(L)}_{K,M}\left(\frac{Ku_s}{N}, \frac{Kv_s}{N}\right) = W^{(L)}_{K,M}\left(\frac{K\lceil x_s r_n\rceil}{N}, \frac{K\lceil y_s r_n\rceil}{N}\right) := \hat{W}^{(L)}_{n,K,M}(x_s, y_s). \tag{3.66}$$

Now, let $\{w_1, w_2, \dots, w_{|V(H)|}\}$ be any labelling of the vertices in $V(H)$ and z_j be such that $w_j = \lceil z_j r_n\rceil$. Then, as in (3.29), using (3.65) and (3.66), for every graph H with at most a edges and at most b isolated vertices and every vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_{|V(H)|})$, (3.64) can be rewritten as,

$$\begin{aligned}
&\mathbb{E}\big[\big(\overline{T}^+_{n,L,K,M}\big)^a\big(\overline{Y}_{n,L,K,M}\big)^b\big] \\
&\quad = \sum_{H\in\mathscr{G}_{a,b}} \sum_{\boldsymbol{\eta}\in[0,b]^{|V(H)|}} c(H, \boldsymbol{\eta}) \int_{\mathscr{B}_{K,n}} t\big(H, \hat{W}^{(L)}_{n,K,M}, \boldsymbol{z}\big) \prod_{j=1}^{|V(H)|} \hat{\Delta}^{(L)}_{n,K,M}(z_j)^{\eta_j}\,\mathrm{d}z_j,
\end{aligned} \tag{3.67}$$

where, as in (3.33), $\mathscr{B}_{K,n} := [0, \frac{\lceil Kr_n\rceil}{r_n}]^{|V(H)|}$, $\boldsymbol{z} = (z_1, z_2, \dots, z_{|V(H)|})$, $t(H, \cdot, \boldsymbol{z})$ is as defined in Definition 3.1, and $\mathscr{G}_{a,b}$ is the collection of graphs with at most a edges and at most b isolated vertices, and $[0, b]^{|V(H)|} := \{0, 1, 2, \dots, b\}^{|V(H)|}$.

Now, since the sum in (3.67) is over a finite set (not depending on n) and each term in the integrand is bounded (by a function of H, K, and M), the integral over $\mathscr{B}_{K,n}$ can be replaced by the integral over $\mathscr{B}_K = [0, K]^{|V(H)|}$, as $n \to \infty$. Moreover, note that the functions $\hat{\Delta}^{(L)}_{n,K,M}$ and $\hat{W}^{(L)}_{n,K,M}$ converge in $L_1([0, K])$ and $L_1([0, K]^2)$ to $\Delta^{(L)}_{K,M}$ and $W^{(L)}_{K,M}$ respectively, as $n \to \infty$. Then using a telescoping argument as in the proof of Lemma 3.6, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\big[\big(\overline{T}^+_{n,L,K,M}\big)^a\big(\overline{Y}_{n,L,K,M}\big)^b\big] \\
&\quad = \sum_{H\in\mathscr{G}_{a,b}} \sum_{\boldsymbol{\eta}=(\eta_1,\eta_2,\ldots,\eta_{|V(H)|})\in[0,b]^{|V(H)|}} c(H,\boldsymbol{\eta}) \int_{\mathscr{B}_K} t\big(H, W^{(L)}_{K,M}, \boldsymbol{z}\big) \prod_{j=1}^{|V(H)|} \Delta^{(L)}_{K,M}(z_j)^{\eta_j}\,\mathrm{d}z_j.
\end{aligned}$$

Next, recall that $\Delta^{(L)}_{K,M}$ and $W^{(L)}_{K,M}$ converge in $L_1([0, K])$ and $L_1([0, K]^2)$ to $\Delta_{K,M}$ and $W_{K,M}$ respectively, as $L \to \infty$, and so we have

$$\begin{aligned}
&\lim_{L\to\infty}\lim_{n\to\infty} \mathbb{E}\big[\big(\overline{T}^+_{n,L,K,M}\big)^a\big(\overline{Y}_{n,L,K,M}\big)^b\big] \\
&\quad = \sum_{\substack{H\in\mathscr{G}_{a,b}\\ \boldsymbol{\eta}\in[0,b]^{|V(H)|}}} c(H,\boldsymbol{\eta}) \int_{\mathscr{B}_K} t(H, W_{K,M}, \boldsymbol{z}) \prod_{j=1}^{|V(H)|} \Delta_{K,M}(z_j)^{\eta_j}\,\mathrm{d}z_j \\
&\quad = \sum_{\substack{H\in\mathscr{G}_{a,b}\\ \boldsymbol{\eta}\in[0,b]^{|V(H)|}}} c(H,\boldsymbol{\eta}) \int_{\mathscr{B}_K} t(H, W_K, \boldsymbol{z})
\end{aligned}$$

(recall (3.46) and (3.48))

$$\begin{aligned}
&\qquad \times \prod_{j=1}^{|V(H)|} \Big(d_K(z_j) - \int_0^K W_K(z_j, v)\,\mathrm{d}v\Big)^{\eta_j} \chi_{K,M}(z_j)\,\mathrm{d}z_j \\
&\quad = \lim_{n\to\infty} \mathbb{E}\big[\big(T^+_{n,K,M}\big)^a\big(Y_{n,K,M}\big)^b\big], \qquad (3.68)
\end{aligned}$$

where $\chi_{K,M}(z_j) := \mathbf{1}\{d_K(z_j) \le M\}$ and the last step follows by combining (3.33) and Lemma 3.6.

The equality of the limiting joint moments in (3.68) and Lemma 3.8, implies, by a diagonalization argument, that for every fixed $K, M \ge 1$ and $t_1, t_2 > 0$, we can find sequences n_j and L_j both increasing to $+\infty$ as $j \to \infty$, such that

$$\lim_{j\to\infty} \mathbb{E}\big[\big(\overline{T}^+_{n_j,L_j,K,M}\big)^a\big(\overline{Y}_{n_j,L_j,K,M}\big)^b\big] = \lim_{n\to\infty} \mathbb{E}\big[\big(T^+_{n,K,M}\big)^a\big(Y_{n,K,M}\big)^b\big] =: \mu_{a,b}, \tag{3.69}$$

for all nonnegative integers a, b, and,

$$\begin{aligned}
&\lim_{j\to\infty} \mathbb{E}\exp\big\{-t_1\overline{T}^+_{n_j,L_j,K,M} - t_2\overline{Y}_{n_j,L_j,K,M}\big\} \\
&\quad = \mathbb{E}\exp\bigg\{\frac{1}{2}\int_0^K\int_0^K \phi_{t_1,K,M}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y) - t_2\int_0^K \Delta_{K,M}(x)\,\mathrm{d}N(x)\bigg\}.
\end{aligned} \tag{3.70}$$

By Lemma 3.7 and (3.69), $\mu_{a,b} = \mathbb{E}[(T^+_{K,M})^a(Y_{K,M})^b]$, and these mixed moments satisfy the Carleman moment condition. Hence, by (3.69), $(\overline{T}^+_{n_j,L_j,K,M}, \overline{Y}_{n_j,L_j,K,M}) \xrightarrow{D} (T^+_{K,M}, Y_{K,M})$ as $j \to \infty$. The result in (3.62) then follows from (3.70).

For (3.63) we compute the joint moment generating function of $(T^+_{n,K,M}, Z_{n,K,M})$:

$$\begin{aligned}
&\mathbb{E}[\exp\{-t_1 T^+_{n,K,M} - t_2 Z_{n,K,M}\}|(X_v)_{v\in V^+_{n,K,M}}] \\
&\quad = \exp\{-t_1 T^+_{n,K,M}\}\mathbb{E}[\exp\{-t_2 Z_{n,K,M}\}|(X_v)_{v\in V^+_{n,K,M}}] \\
\text{(recall (3.26))}\quad &\quad = \exp\{-t_1 T^+_{n,K,M}\}(1-p_n(1-e^{-t_2}))^{\sum_{u\in V^+_{n,K,M}} X_u d_u^{\pm}}.
\end{aligned}$$

Hence, using $(T^+_{n,K,M}, Y_{n,K,M}) \xrightarrow{D} (T^+_{K,M}, Y_{K,M})$, as $n\to\infty$, and the dominated convergence theorem,

$$\begin{aligned}
\mathbb{E}[\exp\{-t_1 T^+_{n,K,M} - t_2 Z_{n,K,M}\}] &= \mathbb{E}[\exp\{-t_1 T^+_{n,KM}\}(1-p_n(1-e^{-t_2}))^{\frac{Y_{n,K,M}}{p_n}}] \\
&\to \mathbb{E}[\exp\{-t_1 T^+_{K,M}\}\exp\{Y_{K,M}(e^{-t_2}-1)\}].
\end{aligned} \tag{3.71}$$

Note that, by (3.62), the RHS of (3.71) is the moment generating function of $(T^+_{K,M}, Y_{K,M})$ evaluated at the points $-t_1$ and $-(1-e^{t_2})$. This implies, $(T^+_{n,K,M}, Z_{n,K,M}) \to (T^+_{K,M}, Z_{K,M})$ in distribution and in moments (by uniform integrability, using Lemma 3.4), where the joint moment generating function is given by (3.63). □

3.6. *Completing the proof of* (1.13) *in Theorem* 1.1. We now combine the results from the previous sections and complete the proof of (1.13).

LEMMA 3.10. *Fix $M>0$ large enough. Under the assumptions of Theorem* 1.1, *$(T^+_{n,K,M}, Z_{n,K,M})$ converges to (T^+_M, Z_M) under the double limit as $n\to\infty$ followed by $K\to\infty$, in distribution and in moments, where the limiting moment generating function is given by*

$$\begin{aligned}
&\mathbb{E}\exp\{-t_1 T^+_M - t_2 Z_M\} \\
&\quad = \mathbb{E}\exp\left\{\frac{1}{2}\int_0^\infty\int_0^\infty \phi_{t_1,M}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y) - \hat{t}_2\int_0^\infty \Delta_M(x)\,\mathrm{d}N(x)\right\},
\end{aligned} \tag{3.72}$$

for $t_1, t_2 > 0$, where

- $\hat{t}_2 := 1-e^{-t_2}$
- $\phi_{t_1,M}(x,y) := \log(1-W^{(M)}(x,y)+W^{(M)}(x,y)e^{-t_1})$, *where* $W^{(M)}(x,y) = W(x,y)\times \mathbf{1}\{d(x)\le M, d(y)\le M\}$, *with* $W:[0,\infty)^2\to[0,1]$ *and* $d:[0,\infty)\to[0,\infty)$ *as in the statement of Theorem* 1.1, *and*
- $\Delta_M(x) := (d(x) - \int_0^\infty W(x,y)\,\mathrm{d}y)\mathbf{1}\{d(x)\le M\}$.

PROOF. Let $W:(0,\infty)^2\to[0,1]$ and $d:[0,\infty)\to[0,\infty)$ be as in the statement of Theorem 1.1. Define $W_K(x,y) := W(x,y)\mathbf{1}\{x,y\in[0,K]\}$ and $d_K(x) := d(x)\mathbf{1}\{x\in[0,K]\}$. Then the conditions (1.11) and (1.12) imply that the sequence of functions $\{W_K\}_{K\ge1}$ and $\{d_K\}_{K\ge1}$, satisfy the conditions (3.27) and (3.28), where $\phi_{n,K}$ the identity map from $[0,K]$ to $[0,K]$, for all $n\ge1$. Therefore, by Lemma 3.9, $(T^+_{n,K,M}, Z_{n,K,M})$ converge in distribution and in moments, as $n\to\infty$, to $(T^+_{K,M}, Z_{K,M})$. Thus, to prove the lemma it suffices to compute the limiting distribution of $(T^+_{K,M}, Z_{K,M})$ as $K\to\infty$.

To this effect, using Observation 3.3 below, gives, for any $t_1>0$

$$\begin{aligned}
\int_0^K\int_0^K \phi_{t_1,K,M}(x,y)\,\mathrm{d}x\,\mathrm{d}y &\to \int_0^\infty\int_0^\infty \phi_{t_1,M}(x,y)\,\mathrm{d}x\,\mathrm{d}y, \\
\int_0^K \Delta_{K,M}(x)\,\mathrm{d}x &\to \int_0^\infty \Delta_M(x)\,\mathrm{d}x,
\end{aligned} \tag{3.73}$$

as $K \to \infty$. Also, noting that $\phi_{t_1,M}(x,y) \le \phi_{t_1,K,M}(x,y)$ for all x, y, it follows that $\lim_{K\to\infty} \|\phi_{t_1,K,M} - \phi_{t_1,M}\|_{L_1([0,\infty)^2)} = 0$. Similarly, using (3.73), it can be shown that $\lim_{K\to\infty} \|\Delta_{K,M} - \Delta_M\|_{L_1([0,\infty)^2)} = 0$. This implies (using the convergence of stochastic integrals in Proposition B.2), as $K \to \infty$, that

$$\int_0^K \int_0^K \phi_{t_1,K,M}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y) \xrightarrow{L_1} \int_0^\infty \int_0^\infty \phi_{t_1,M}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y),$$

$$\int_0^K \Delta_{K,M}(x)\,\mathrm{d}N(x) \xrightarrow{L_1} \int_0^\infty \Delta_M(x)\,\mathrm{d}N(x).$$

Therefore, taking limit as $K \to \infty$ in (3.63) we see that the moment generating function converge to the RHS of (3.72) (using the convergence of the stochastic integrals above and the dominated convergence theorem). This shows, $(T^+_{K,M}, Z_{K,M}) \xrightarrow{D} (T^+_M, Z_M)$, as $K \to \infty$, with the joint moment generating function of (T^+_M, Z_M) given by (3.72). To see that this convergence is also in moments, recall from Lemma 3.4 that

$$\limsup_{K\to\infty} \limsup_{n\to\infty} \mathbb{E}\big[(T^+_{n,K,M})^a (Z_{n,K,M})^b\big] \lesssim_{a,b,M} 1.$$

Therefore, by uniform integrability, the convergence in moments follows. $\square$

Combining the results above we can now derive the limiting distribution of $(T^+_{n,K,M}, Z_{n,K,M}, W^-_{n,K})$, as $n \to \infty$ followed by $K \to \infty$.

LEMMA 3.11. *Let (T^+_M, Z_M) be random variables with joint moment generating function as in* (3.72). *Then, under the assumptions of Theorem* 1.1,

$$(T^+_{n,K,M}, Z_{n,K,M}, W^-_{n,K}) \xrightarrow{D} (T^+_M, Z_M, W),$$

in distribution and in all (mixed) moments, as $n \to \infty$ followed by $K \to \infty$, where $W \sim$ $\mathrm{Pois}(\lambda_0)$ *and W is independent of (T^+_M, Z_M).*

PROOF. By (3.63) and (3.72), as $n \to \infty$, followed by $K \to \infty$, for $t_1, t_2 \ge 0$,

$$\begin{aligned}
&\mathbb{E}\big[\exp\{-t_1 T^+_{n,K,M} - t_2 Z_{n,K,M}\}\big] \\
&\quad \to \mathbb{E}\exp\left\{\frac{1}{2}\int_0^\infty \int_0^\infty \phi_{t_1,M}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y) - \hat{t}_2 \int_0^\infty \Delta_M(x)\,\mathrm{d}N(x)\right\},
\end{aligned} \tag{3.74}$$

where $\hat{t}_2 = 1 - e^{-t_2}$. This shows $(T^+_{n,K,M}, Z_{n,K,M}) \xrightarrow{D} (T^+_M, Z_M)$, as $n \to \infty$ followed by $K \to \infty$, with joint moment generating function as above.

Now, recall the definition of $W^-_{n,K}$ from (3.5). By definition, $W^-_{n,K}$ is independent of $(T^+_{n,K,M}, Z_{n,K,M})$ and $W^-_{n,K} \xrightarrow{D} W \sim \mathrm{Pois}(\lambda_0)$ (by condition (a) in Theorem 1.1). Therefore,

$$(T^+_{n,K,M}, Z_{n,K,M}, W^-_{n,K}) \xrightarrow{D} (T^+_M, Z_M, W),$$

as required.

Finally, by Lemma 3.4 $\limsup_{K\to\infty} \limsup_{n\to\infty} \mathbb{E}[(T^+_{n,K,M})^a Z^b_{n,K,M}] \lesssim_{a,b,M} 1$, and by Lemma 3.2 $\limsup_{K\to\infty} \limsup_{n\to\infty} \mathbb{E}[(W^-_{n,K})^c] \lesssim_{c,M} 1$, for all positive integers a, b, c. Therefore, by the Cauchy–Schwarz inequality,

$$\mathbb{E}\big[(T^+_{n,K,M})^a Z^b_{n,K,M} (W^-_{n,K})^c\big] \le \big(\mathbb{E}\big[(T^+_{n,K,M})^{2a} Z^{2b}_{n,K,M}\big]\mathbb{E}\big[(W^-_{n,K})^{2c}\big]\big)^{\frac{1}{2}} \lesssim_{a,b,c,M} 1.$$

Then by uniformly integrability the convergence of the mixed moments follows. $\square$

The above lemma implies that $T^+_{n,K,M} + Z_{n,K,M} + W^-_{n,K} \to T^+_M + Z_M + W$, in distribution and in all moments. Then by Proposition 3.1 (recall (3.2)),

$$T_{n,M} = T^+_{n,K,M} + T^{\pm}_{n,K,M} + T^-_{n,K,M} \to T^+_M + Z_M + W, \tag{3.75}$$

in all moments. Convergence in distribution follows by verifying the Carleman moment condition for $T^+_M + Z_M + W$, as follows:

LEMMA 3.12. *Fix $M \geq 1$. Let (T^+_M, Z_M) be random variables with joint moment generating function as in (3.72). Then, under the assumptions of Theorem 1.1,*

$$T_{n,M} \to T^+_M + Z_M + W,$$

in distribution and in all (mixed) moments, as $n \to \infty$, where $W \sim \mathrm{Pois}(\lambda_0)$ and W is independent of (T^+_M, Z_M).

PROOF. The convergence in moments follows from (3.75). To establish convergence in distribution we need to verify the Carleman moment condition. To this end, let $G_{n,M}$ be the graph obtained from G_n by removing all vertices with degree greater than Mr_n along with all the edges adjacent on them. Then observe that, for $a \geq 1$,

$$\mathbb{E}T^a_{n,M} = \sum_{(u_1,v_1),(u_2,v_2),\dots,(u_a,v_a)\in E(G_{n,M})} \frac{1}{r_n^{|V(H)|}} \leq a^a \sum_{H\in\mathcal{H}_a} \frac{N(H, G_{n,M})}{r_n^{|V(H)|}}, \tag{3.76}$$

where H is the graph formed by the union of the edges $(u_1, v_1), (u_2, v_2), \dots, (u_a, v_a)$, and $\mathcal{H}_a$ the collection of all nonisomorphic graphs with at most a edges and no isolated vertices.

Now, using $N(H, G_{n,M}) \leq |E(G_n)|^{\nu(H)}(Mr_n)^{|V(H)|-2\nu(H)}$, where $\nu(H)$ is the number of connected components of H, and (1.6), it follows that there exists some constant $C_1 > 0$ such that, for n large enough, $\frac{N(H,G_{n,M})}{r_n^{|V(H)|}} \leq C_1^a M^{|V(H)|-2\nu(H)} \leq C_1^a M^{2a}$, since $|V(H)| \leq 2a$ and $\nu(H) \leq a$, for $H \in \mathcal{H}_a$. Finally, using $|\mathcal{H}_a| \leq C^a a^{a+1} \leq (2C)^a a^a$, for some constant $C > 0$ [5], Theorem 5, we get

$$\mu_a := \lim_{n\to\infty} \mathbb{E}T^a_{n,M} \lesssim C_1^a (2C)^a M^{2a} a^{2a}. \tag{3.77}$$

This shows that

$$\sum_{a=1}^{\infty} \frac{1}{\mu_a^{\frac{1}{2a}}} \gtrsim_M \sum_{a=1}^{\infty} \frac{1}{a} = \infty,$$

which verifies the Carleman moment condition and completes the proof. □

By monotonicity, as $M \to \infty$, there exist random variables (T^+, Z) such that $(T^+_M, Z_M) \xrightarrow{D} (T^+, Z)$, with joint moment generating function (which is obtained by taking the limit as $M \to \infty$ in (3.72) and using Proposition B.2),

$$\begin{aligned}&\mathbb{E}\exp\{-t_1 T^+ - t_2 Z\}\\ &\quad= \mathbb{E}\exp\left\{-\frac{1}{2}\int_0^\infty\int_0^\infty \phi_{W,t_1}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y) - \hat{t}_2\int_0^\infty \Delta(x)\,\mathrm{d}N(x)\right\},\end{aligned} \tag{3.78}$$

where $\phi_{W,t_1}(\cdot,\cdot)$ and $\Delta(\cdot)$ are as defined in the statement of Theorem 1.1. (Note that

$$\int_0^\infty\int_0^\infty \phi_{W,t_1}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y) < \infty \quad \text{and} \quad \int_0^\infty \Delta(x)\,\mathrm{d}N(x) < \infty$$

almost surely, by Observation 3.3 and finiteness of stochastic integrals for L_1 integrable functions.) Thus, $T_M := T^+_M + Z_M + W$ converges in distribution to $T := T^+ + Z + W$, as $M \to$

∞, where $W \stackrel{D}{=} \mathrm{Pois}(\lambda_0)$ is independent of (T^+, Z). Therefore, using $T_n = T_{n,M} + o_P(1)$, where the $o_P(1)$-term goes to zero as $n \to \infty$ followed by $M \to \infty$ (recall Lemma 3.1), and Lemma 3.12, it follows that $T_n \xrightarrow{D} T^+ + Z + W$, where $W \sim \mathrm{Pois}(\lambda_0)$, W is independent of (T^+, Z), and the joint moment generating function of (T^+, Z) is given by (3.78). This completes the proof of (1.14).

The finiteness of the integrals of W and d, required in the proof above, is established below:

OBSERVATION 3.3. *With $W(\cdot,\cdot)$, $d(\cdot)$, ϕ_{t_1} as in the statement of Theorem 1.1, the following hold*:

(a) $\int_0^\infty \int_0^\infty W(x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty$,
(b) $\int_0^\infty \int_0^\infty |\phi_{W,t_1}(x,y)|\,\mathrm{d}x\,\mathrm{d}y < \infty$,
(c) $\int_0^\infty d(x)\,\mathrm{d}x < \infty$.

PROOF. Fixing $K \geq 1$, gives $\frac{1}{r_n^2}|E(G_n)| \geq \frac{1}{r_n^2}|E(G_{n,K}^+)| = \frac{1}{2}\int_{[0,K]^2} W_{G_n}(x,y)\,\mathrm{d}x\,\mathrm{d}y$, which on letting $n \to \infty$ along with assumption (1.11), gives

$$\int_{[0,K]^2} W(x,y)\,\mathrm{d}x\,\mathrm{d}y \lesssim \limsup_{n\to\infty} \frac{1}{r_n^2}\big|E(G_n)\big|.$$

Since this holds for every $K \geq 1$, letting $K \to \infty$ along with monotone convergence theorem gives

$$\int_{[0,\infty)^2} W(x,y)\,\mathrm{d}x\,\mathrm{d}y \lesssim \limsup_{n\to\infty} \frac{|E(G_n)|}{r_n^2} = O(1),$$

by (1.6). This completes the proof of (a).

The conclusion in part (b) follows immediately by invoking part (a) and noting that $0 \leq -\phi_{W,t_1}(x,y) \lesssim_{t_1} W(x,y)$.

To show (c), note that by condition (1.12), for K, M large enough,

$$\int_0^K d(x)\mathbf{1}\{d(x) \leq M\}\,\mathrm{d}x = \lim_{n\to\infty}\int_0^K d_{W_{G_n}}(x)\mathbf{1}\{d_{W_{G_n}}(x) \leq M\}\,\mathrm{d}x \lesssim \limsup_{n\to\infty} \frac{|E(G_n)|}{r_n^2}.$$

Taking limit $K \to \infty$ followed by $M \to \infty$ on both sides, gives

$$\int_0^\infty d(x)\,\mathrm{d}x \lesssim \limsup_{n\to\infty} \frac{|E(G_n)|}{r_n^2},$$

from which the desired conclusion follows on using (1.6). □

4. Proof of Theorem 1.2. We begin by recalling that $\mathcal{W}_K$ is the set of all symmetric measurable functions from $[0,K]^2 \to [0,1]$. Denote by $\mathcal{M}_K$ the set of all measure preserving bijections ϕ from $[0,K] \to [0,K]$. Moreover, for any function $\phi \in \mathcal{M}_K$, let $W^\phi(x,y) = W(\phi(x),\phi(y))$ and $f^\phi(x) = f(\phi(x))$, for $W \in \mathcal{W}_K$ and $f : [0,K] \to [0,M]$. The following proposition shows the joint sequential compactness of the cut-metric and the L_1 distance.

PROPOSITION 4.1. *Fix $K, M \geq 1$. Then given a sequence of measurable functions $W_n \in \mathcal{W}_K$ and a sequence of measurable functions $f_n : [0,K] \to [0,M]$, there exists a subsequence $\{n_s\}_{s\geq 1}$ such that*,

$$\lim_{s\to\infty}\inf_{\phi\in\mathcal{M}_K}\big\{\big\|W_{n_s}^\phi - W_K\big\|_{\square([0,K]^2)} + \big\|f_{n_s}^\phi - f_K\big\|_{L_1([0,K])}\big\} = 0,$$

for some $W_K \in \mathcal{W}_K$ and $f_K \in L_1([0,K])$.

The proof of the proposition is given below in Section 4.1. First, we use it to complete the proof of Theorem 1.2. To this end, suppose (1.6) holds and T_n converges in distribution to a random variable T. Begin by labeling the vertices of G_n in nonincreasing order of the degrees. Now, fix $M \in \mathcal{D}$ (as in Definition 3.2) and recall the definition of $T_{n,M}$ from (1.17), and use Lemma 3.1 to note that $T_{n,M} \xrightarrow{D} T$, under the double limit as $n \to \infty$ followed by $M \to \infty$. Next, fix $K \geq 1$ and recall from (3.2),

$$T_{n,M} = T^{+}_{n,K,M} + T^{\pm}_{n,K,M} + T^{-}_{n,K,M}. \tag{4.1}$$

We will now proceed to find a subsequence $\{n_s\}_{s\geq 1}$ along which the RHS above will have a limiting distribution in the form (1.13).

To begin with, observe that $\int_K^\infty \int_K^\infty W_{G_n}(x,y)\,\mathrm{d}x\,\mathrm{d}y \lesssim \frac{|E(G_n)|}{r_n^2} \lesssim 1$ by (1.6), for n large enough. Therefore, for every $K \geq 1$ fixed, there exists a subsequence depending on K such that

$$\lambda_0(K) := \lim_{n\to\infty} \frac{1}{2} \int_K^\infty \int_K^\infty W_{G_n}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

exists along that subsequence. Therefore, refining the subsequences at every stage and by a diagonalization argument, there exists a common subsequence $\{n_s\}_{s\geq 1}$ along which

$$\lim_{s\to\infty} \frac{1}{2} \int_K^\infty \int_K^\infty W_{G_{n_s}}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lambda_0(K),$$

for every $K \geq 1$. Now, note that

$$\begin{aligned} \lambda_0(K+1) &= \lim_{s\to\infty} \frac{1}{2} \int_{K+1}^\infty \int_{K+1}^\infty W_{G_{n_s}}(x,y) \leq \lim_{s\to\infty} \frac{1}{2} \int_K^\infty \int_K^\infty W_{G_{n_s}}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \lambda_0(K), \end{aligned}$$

which implies

$$\lambda_0 := \lim_{K\to\infty} \lambda_0(K) = \lim_{K\to\infty} \lim_{s\to\infty} \frac{1}{2} \int_K^\infty \int_K^\infty W_{G_{n_s}}(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{4.2}$$

exists.

Next, applying Proposition 4.1 on the functions

$$W_{G_n,K,M}(x,y) := W_{G_n}(x,y)\mathbf{1}\{x, y \in [0,K], d_{W_{G_n,K}}(x) \leq M, d_{W_{G_n,K}}(y) \leq M\}$$

and

$$d_{W_{G_n},K}(x|M) := d_{W_{G_n}}(x)\mathbf{1}\{x \in [0,K], d_{W_{G_n,K}}(x) \leq M\},$$

gives a sequence of functions $\phi_{n,K} \in \mathcal{M}_K$ and a subsequence $\{n_s\}_{s\geq 1}$ such that,

$$\lim_{s\to\infty} \| W^{\phi_{n_s,K}}_{G_{n_s},K,M} - W_{K,M} \|_{\square([0,K]^2)} \quad \text{and} \quad \| d^{\phi_{n_s,K}}_{W_{G_{n_s}},K,M}(\cdot|M) - d_{K,M} \|_{L_1([0,K])} = 0,$$

for some $W_{K,M} \in \mathcal{W}_K$ and $d_{K,M} \in L_1([0,K])$. This shows that, along this subsequence the assumptions of Lemma 3.5 are satisfied, therefore, by Lemma 3.9, along this subsequence

$$T^{+}_{n_s,K,M} + Z_{n_s,K,M} \to J_{1,K,M} + J_{2,K,M}, \tag{4.3}$$

in distribution and in moments, where the joint moment generating function of $(J_{1,K,M}, J_{2,K,M})$ is given by: For $t_1, t_2 \geq 0$,

$$\begin{aligned} &\mathbb{E}\exp\{-t_1 J_{1,K,M} - t_2 J_{2,K,M}\} \\ &\quad = \mathbb{E}\exp\left\{\frac{1}{2}\int_0^\infty \int_0^\infty \phi_{W_K,t_1}(x,y)\,\mathrm{d}N(x)\,\mathrm{d}N(y) - \hat{t}_2 \int_0^\infty \Delta_{K,M}(x)\,\mathrm{d}N(x)\right\}, \end{aligned}$$

with

– $\hat{t}_2 = 1 - e^{-t_2}$,
– $\phi_{W_{K,M},t_1}(x, y) := \log(1 - W_{K,M}(x, y) + W_{K,M}(x, y)e^{-t_1})$,
– $\Delta_{K,M}(x) := d_{K,M}(x) - \int_0^\infty W_{K,M}(x, y)\,\mathrm{d}y$.

Now, by the convergence in moments and Lemma 3.4, for every integer $r \geq 1$,

$$\mathbb{E}[(J_{1,K,M} + J_{2,K,M})^r] = \lim_{s\to\infty} \mathbb{E}[(T^+_{n_s,K,M} + T^\pm_{n_s,K,M})^r] \lesssim_{r,M} 1.$$

Therefore, there exists a subsequence $\{K_j\}_{j\geq 1}$ such that as $j \to \infty$, $J_{1,K_j,M} + J_{2,K_j,M} \to J_M$, for some random variable $J_M \in \overline{\mathcal{P}}(\mathcal{W}, \mathcal{F})$ (recall Definition 1.2), in distribution and in moments. Therefore, refining the subsequence in (4.2) and (4.3), and using the independence of $T^+_{n_s,K_j,M} + Z_{n_s,K_j,M}$ and $W^-_{n_s,K_j}$, it follows that, as $s \to \infty$ followed by $j \to \infty$,

$$T^+_{n_s,K_j,M} + Z_{n_s,K_j,M} + W^-_{n_s,K_j} \to J_M + J_0,$$

in distribution and in moments, where $J_0 \sim \mathrm{Pois}(\lambda_0)$ and J_0 independent of J_M, for all $M \in \mathcal{D}$. Then by the proof of Lemma 3.12, it follows that as, $s \to \infty$ followed $j \to \infty$,

$$T^+_{n_s,K_j,M} + T^\pm_{n_s,K_j,M} + T^-_{n_s,K_j,M} \xrightarrow{D} J_M + J_0.$$

Now, as $T_{n_s,M} \xrightarrow{D} T$, when $s \to \infty$ followed by $M \to \infty$, recalling (4.1) we get, $J_M + J_0 \xrightarrow{D} J + J_0 \overset{D}{=} T$, as $M \to \infty$, where J is independent of J_0 and $J \in \overline{\mathcal{P}}(\mathcal{W}, \mathcal{F})$, since $J_M \in \overline{\mathcal{P}}(\mathcal{W}, \mathcal{F})$.

4.1. *Proof of Proposition* 4.1. Without loss of generality, we assume $K = M = 1$. Hereafter, we fix $L \geq 1$. Then, we have the following:

- For the graphon $W_n \in \mathcal{W}_1$ by the weak regularity lemma [8], Corollary 3.4, we can find a partition $\Pi_{n,L} = \{\pi_{n,L}(i)\}_{i\in[q_L]}$ of [0, 1] into measurable sets, with $q_L \lesssim_L 1$ (a constant depending only on L), such that

$$\|W_n - W_{n,L}\|_{\square([0,1]^2)} \leq \frac{1}{L}, \tag{4.4}$$

  where

$$W_{n,L}(x, y) = b_{W_n,L}(i, j) = \frac{1}{\lambda(\pi_i)\lambda(\pi_j)} \int_{\pi_i\times\pi_j} W_n(x, y)\,\mathrm{d}x\,\mathrm{d}x,$$

  for $x \in \pi_i$ and $y \in \pi_j$. (Here, $\lambda(A)$ denotes the Lebesgue measure of a measurable set $A \subset [0, 1]$.) Moreover, the partitions can be constructed in such a way that $\Pi_{n,L+1} = \{\pi_{n,L+1}(i)\}_{a\in[q_{L+1}]}$ is a refinement of $\Pi_{n,L}$ (by [8], Corollary 3.4).
- Similarly, for the function f_n, there exists a partition $\Gamma_{n,L} = \{\gamma_{n,L}(i)\}_{i\in[r_L]}$ of [0, 1] into $r_L \lesssim_L 1$ (a constant depending only on L) measurable sets and a vector $z_{f_n,L} = (z_{f_n,L}(i))_{i\in[r_L]}$ with entries in [0, 1], such that the function

$$f_{n,L}(x) := z_{f_n,L}(i) := \frac{1}{\lambda(\gamma_{n,L}(i))} \int_{\gamma_{n,L}(i)} f_n(x)\,\mathrm{d}x \quad \text{if } x \in \gamma_{n,L}(i), \tag{4.5}$$

  satisfies

$$\|f_n - f_{n,L}\|_{L_1([0,1])} \leq \frac{1}{L}. \tag{4.6}$$

  Moreover, as before, the partitions can be constructed in such a way that $\Gamma_{n,L+1} = \{\gamma_{n,L+1}(i)\}_{a\in[r_{L+1}]}$ is a refinement of $\Gamma_{n,L}$.

Given the partitions $\Pi_{n,L} = \{\pi_{n,L}(i)\}_{i\in[q_L]}$ and $\Gamma_{n,L} = \{\gamma_{n,L}(i)\}_{i\in[r_L]}$, the class of sets

$$\{\theta_{n,L}(i_1, i_2) := \pi_{n,L}(i_1) \cap \gamma_{n,L}(i_2)\}_{i_1\in[q_L],i_2\in[r_L]},$$

forms a partition of $[0, 1]$, which refines both the partitions $\Pi_{n,L}$ and $\Gamma_{n,L}$ (with possibly some empty sets). Relabel the sets $\{\theta_{n,L}(i_1, i_2)\}_{i_1\in[q_L],i_2\in[r_L]}$ by $\{\theta_{n,L}(i)\}_{i\in[q_L r_L]}$ by taking a bijection from $[q_L] \times [r_L] \to [q_L r_L]$, and denote this partition of $[0, 1]$ by $\Theta_{n,L} := \{\theta_{n,L}(i)\}_{i\in[q_L r_L]}$. Now, setting $\beta_{n,L}(i) := \lambda(\theta_{n,L}(i))$, there exists a measure preserving bijection $\phi_{n,L} : [0, 1] \to [0, 1]$ such that the interval

$$\left(\sum_{i=1}^{a-1} \beta_{n,L}(i), \sum_{i=1}^{a} \beta_{n,L}(i)\right] \quad \text{maps to the set } \theta_{n,L}(a), \text{ for each } 1 \le a \le q_L r_L.$$

Thus, the functions $W_{n,L}^{\phi_{n,L}}$ and $f_{n,L}^{\phi_{n,L}}$ are both step functions on $[0, 1]^2$ and $[0, 1]$ with intervals and rectangles as steps, respectively. Then, we can find a common subsequence $\{n_s\}_{s\ge1}$ along which the sequence of vectors

$$\left(\{\beta_{n_s,L}(i)\}_{i\in[q_L r_L]}, \boldsymbol{B}_{W_{n_s},L}, \boldsymbol{z}_{f_{n_s},L}\right) \in [0, 1]^{q_L r_L + q_L^2 + r_L}$$

converge. (Here, we consider $\boldsymbol{B}_{W_{n_s},L}$ as a vector of length q_L^2.) In particular, this means that along this subsequence the functions $W_{n_s,L}^{\phi_{n,L}}$ and $f_{n_s,L}^{\phi_{n,L}}$ converge almost surely to step functions $W_L : [0, 1]^2 \to [0, 1]$ and $f_L : [0, 1] \to [0, 1]$, respectively.

Now, let $(U, V) \sim \mathrm{Unif}([0, 1]^2)$, and let $\mathcal{F}_L$ denote the sub-sigma algebra of $\mathscr{B}([0, 1]^2)$ (the Borel sigma algebra on $[0, 1]^2$) generated by the collection

$$\left\{\mathbf{1}\left\{U \in \left(\sum_{j=1}^{i_1-1} \beta_L(j), \sum_{j=1}^{i_1} \beta_L(j)\right]\right\}, \mathbf{1}\left\{V \in \left(\sum_{j=1}^{i_2-1} \beta_L(j), \sum_{j=1}^{i_2} \beta_L(j)\right]\right\}, i_1, i_2 \in [q_L r_L]\right\},$$

where $\beta_L(i) = \lim_{s\to\infty} \beta_{n_s,L}(i)$ for $i \in [q_L r_L]$. Since the partition

$$\Theta_{n,L+1} = \{\theta_{n,L+1}(i)\}_{i\in[q_{L+1} r_{L+1}]}$$

is a refinement of $\Theta_{n,L} = \{\theta_{n,L}(i)\}_{i\in[q_L r_L]}$, it follows that $\{\mathcal{F}_L\}_{L\ge1}$ is a filtration. Also, the construction implies that for any $(x, y) \in (0, 1]^2$ such that

$$(x, y) \in \left(\sum_{j=1}^{i_1-1} \beta_L(j), \sum_{j=1}^{i_1} \beta_L(j)\right] \times \left(\sum_{j=1}^{i_2-1} \beta_L(j), \sum_{j=1}^{i_2} \beta_L(j)\right] \quad \text{where } 1 \le i_1, i_2 \le q_L r_L,$$

we have

$$\begin{aligned} &W_L(x, y) \\ &\quad = \mathbb{E}\left(W_{L+1}(U, V)\middle|(U, V) \in \left(\sum_{j=1}^{i_1-1} \beta_L(j), \sum_{j=1}^{i_1} \beta_L(j)\right] \times \left(\sum_{j=1}^{i_2-1} \beta_L(j), \sum_{j=1}^{i_2} \beta_L(j)\right]\right), \end{aligned}$$

and

$$f_L(x) = \mathbb{E}\left(f_{L+1}(U)\middle|U \in \left(\sum_{j=1}^{i_1-1} \beta_L(j), \sum_{a=1}^{i_1} \beta_L(j)\right]\right).$$

Thus, both W_L and f_L are bounded martingales with respect to the filtration $\{\mathcal{F}_L\}_{L\ge1}$, and so they converge almost surely and in L_1 to functions W_∞ and f_∞, as $L \to \infty$, respectively. Therefore, by the triangle inequality,

$$\inf_{\phi\in\mathcal{M}_1} \{\|W_{n_s}^{\phi} - W_\infty\|_{\square([0,1]^2)} + \|f_{n_s}^{\phi} - f_\infty\|_{L_1([0,1])}\} \le S_1 + S_2 + S_3, \tag{4.7}$$

where S_1, S_2, S_3 are defined as follows:

$$S_1 := \|W_{n_s}^{\phi_{n_s,L}} - W_{n_s,L}^{\phi_{n_s,L}}\|_{\square([0,1]^2)} + \|f_{n_s}^{\phi_{n_s,L}} - f_{n_s,L}^{\phi_{n_s,L}}\|_{L_1([0,1])} \leq \frac{2}{L},$$

where the last inequality uses (4.4) and (4.6). Next,

$$\begin{aligned} S_2 &:= \|W_{n_s,L}^{\phi_{n_s,L}} - W_L\|_{\square([0,1]^2)} + \|f_{n_s,L}^{\phi_{n_s,L}} - f_L\|_{L_1([0,1])} \\ &\leq \|W_{n_s,L}^{\phi_{n_s,L}} - W_L\|_{L_1([0,1]^2)} + \|f_{n_s,L}^{\phi_{n_s,L}} - f_L\|_{L_1([0,1])}, \end{aligned}$$

which goes to zero as $s \to \infty$, using the fact that $W_{n_s,L}^{\phi_{n_s,L}}$ and $f_{n_s,L}^{\phi_{n_s,L}}$ converges in L_1 to W_L and f_L, respectively. Finally,

$$\begin{aligned} S_3 &:= \|W_L - W_\infty\|_{\square([0,1]^2)} + \|f_L - f_\infty\|_{L_1([0,1])} \\ &\leq \|W_L - W_\infty\|_{L_1([0,1]^2)} + \|f_L - f_\infty\|_{L_1([0,1])}, \end{aligned}$$

which goes to zero as $L \to \infty$, using $W_L \stackrel{L_1}{\to} W$ and $f_L \stackrel{L_1}{\to} f$. Putting together the above three bounds with (4.7), and taking limit as $s \to \infty$ followed by $L \to \infty$, the result follows.

5. Proofs of corollaries.

In this section we prove Corollaries 1.3, 1.4, and 1.5.

5.1. *Proof of Corollary* 1.3. As $\{G_n\}_{n\geq 1}$ is a sequence of dense graphs, assumption (1.6) implies that $r_n = 1/p_n > Cn$, for some constant $C > 0$, when n is large enough. Therefore, by the definition in (1.7), W_{G_n} is zero outside the box $[0,a]^2$, where $a := 1/C$. Hence,

$$\lim_{K\to\infty} \lim_{n\to\infty} \frac{1}{2} \int_K^\infty \int_K^\infty W_{G_n}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{5.1}$$

We begin by showing that W vanishes Lebesgue almost everywhere outside the rectangle $[0,a]^2$. To see this, let $f(x) = \mathbf{1}\{x \geq a\}$ and $g(x) = \mathbf{1}\{x \leq a\}$ for all $x \geq 0$. Fix $L > 0$, and observe that:

$$\begin{aligned} \int_a^{a+L} \int_0^a W(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{[0,a+L]^2} \big(W(x,y) - W_{G_n}(x,y)\big) f(x) g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \|W - W_{G_n}\|_{\square([0,a+L]^2)}. \end{aligned} \tag{5.2}$$

Since the RHS of (5.2) converges to 0 and $W \geq 0$, W must vanish Lebesgue almost everywhere on the rectangle $[a, a+L] \times [0,a]$. This means, since $L > 0$ is arbitrary, W vanishes Lebesgue almost everywhere on $[a,\infty) \times [0,a]$. Interchanging the roles of f and g, it follows that W vanishes Lebesgue almost everywhere on $[0,a] \times [a,\infty)$. Finally, taking $f(x) = g(x) = \mathbf{1}\{x \geq a\}$, for all $x \geq 0$, and proceeding as above, we can show that W vanishes Lebesgue almost everywhere on $[a,\infty) \times [a,\infty)$, as desired.

Now, fix $K \geq a$ such that $\|W_{G_n} - W\|_{\square([0,K]^2)} \to 0$. Next, let $d_W(x) = \int_0^\infty W(x,y)\,\mathrm{d}y = \int_0^a W(x,y)\,\mathrm{d}y$. Then, we have

$$\lim_{n\to\infty} \int_0^K \big(d_{W_{G_n}}(x) - d_W(x)\big)^2\,\mathrm{d}x \to 0.$$

This is because, $\|W_{G_n} - W\|_{\square([0,K]^2)} \to 0$ implies that

$$\int_0^K d_{W_{G_n}}(x)^2\,\mathrm{d}x \to \int_0^K d_W(x)^2\,\mathrm{d}x \quad \text{and} \quad \int_0^K d_{W_{G_n}}(x) d_W(x)\,\mathrm{d}x \to \int_0^K d_W(x)^2\,\mathrm{d}x.$$

Then, by the Cauchy–Schwarz inequality, $\|d_{W_{G_n}} - d_W\|_{L_1([0,K])} \to 0$, for all $K \geq a$ such that $\|W_{G_n} - W\|_{\square([0,K]^2)} \to 0$. This shows, condition (1.12) holds with $d = d_W$, for $M \in \mathcal{D}$

(recall Definition 3.2). Therefore, $\Delta(x) := d_W(x) - \int_0^\infty W(x,y)\,\mathrm{d}y = 0$, and the result in (1.16) follows from Theorem 1.1, with $\Delta = 0$ and $\lambda_0 = 0$ (by (5.1)).

To show (b), note that for any sequence of dense graphs $\{G_n\}_{n\geq 1}$, there exists a constant $a > 0$ and a subsequence $\{n_j\}_{j\geq 1}$ along which $\lim_{j\to\infty} \delta_{\square([0,a]^2)}(W_{G_{n_j}}, W) = 0$, for some $W \in \mathcal{W}_a$ (by the compactness of the metric $\delta_{\square([0,a]^2)}$ in the space $\mathcal{W}_a$, the space of all symmetric measurable functions from $[0,a]^2 \to [0,1]$ [8], Proposition 3.6). This implies, recalling (1.10), there exists a sequence of measure preserving bijections $\{\phi_{n_j}\}_{j\geq 1}$ such that $\lim_{j\to\infty} \|W_{G_{n_j}}^{\phi_{n_j}} - W\|_{\square([0,a]^2)} = 0$. Therefore, by part (a) (recall the discussion in the second item in Remark 1.2 and Lemma 3.5), along this subsequence $T_{n_j} \xrightarrow{D} Q_1$, where the moment generating function of Q_1 is as given in (1.16). This implies $T_n \xrightarrow{D} Q_1$, since, by assumption, T_n converges in distribution, as required.

5.2. *Proof of Corollary* 1.4. Note that $(c) \Rightarrow (a)$ is immediate from Theorem 1.1. Hence, it suffices to show that $(b) \Rightarrow (c)$ and $(a) \Rightarrow (b)$.

We begin with the proof of $(b) \Rightarrow (c)$. Denote by $K_{1,2}$, the two-star graph and let $V_M := \{v \in G_n : d_v \leq M r_n\}$. Then

$$\operatorname{Var}(T_{n,M}) = (1 - p_n^2)\mathbb{E}T_{n,M} + 2p_n^3(1 - p_n)N(K_{1,2}, G_n[V_M]), \tag{5.3}$$

where $N(K_{1,2}, G_n[V_M])$ is the number of two-stars in the subgraph on G_n induced on the vertex set V_M. The conditions

$$\lim_{M\to\infty}\lim_{n\to\infty} \mathbb{E}(T_{n,M}) = \lambda \quad \text{and} \quad \lim_{M\to\infty}\lim_{n\to\infty} \operatorname{Var}(T_{n,M}) = \lambda$$

mean, $\lim_{M\to\infty}\lim_{n\to\infty} p_n^3 N(K_{1,2}, G_n[V_M]) = 0$. Letting

$$d_{W_{G_n[V_M]}}(x) = \int_0^\infty W_{G_n}(x,y)\mathbf{1}\{d_{W_{G_n}}(x) \leq M, d_{W_{G_n}}(y) \leq M\}\,\mathrm{d}y,$$

then gives,

$$\int_0^\infty d_{W_{G_n[V_M]}}(x)^2\,\mathrm{d}x \lesssim p_n^3 N(K_{1,2}, G_n[V_M]) + p_n^3|E(G_{n,M})| \to 0, \tag{5.4}$$

under the same double limit, where we invoke (1.6) to deal with the second term.

We will now verify condition (1.12) in Theorem 1.1. To see this, observe

$$\begin{aligned} d_{W_{G_n}}(x)\mathbf{1}\{d_{W_{G_n}}(x) \leq M\} &= d_{W_{G_n[V_M]}}(x) + \int_0^\infty W_{G_n}(x,y)\mathbf{1}\{d_{W_{G_n}}(y) > M\}\,\mathrm{d}y \\ &\leq d_{W_{G_n[V_M]}}(x) + \frac{1}{M}\int_0^\infty d_{W_{G_n}}(y)\,\mathrm{d}y. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^K d_{W_{G_n}}(x)^2\mathbf{1}\{d_{W_{G_n}}(x) \leq M\}\,\mathrm{d}x \\ &\quad\leq 2\int_0^\infty d_{W_{G_n[V_M]}}(x)^2\,\mathrm{d}x + \frac{2}{M^2}\int_0^K \left(\int_0^\infty d_{W_{G_n}}(y)\,\mathrm{d}y\right)^2 \mathrm{d}x \\ &\quad= 2\int_0^\infty d_{W_{G_n[V_M]}}(x)^2\,\mathrm{d}x + \frac{2K}{M^2}\left(\int_0^\infty d_{W_{G_n}}(y)\,\mathrm{d}y\right)^2. \end{aligned}$$

Now, under the double limit $n \to \infty$ followed by $M \to \infty$, first term in the RHS above goes to 0 by (5.4), and the second term goes to 0 by (1.6). This gives,

$$\lim_{M\to\infty}\lim_{n\to\infty}\int_0^K d_{W_{G_n}}(x)^2\mathbf{1}\{d_{W_{G_n}}(x)\le M\}\,\mathrm{d}x = 0.$$

Therefore, by the Cauchy–Schwarz inequality,

$$\lim_{M\to\infty}\lim_{n\to\infty}\int_0^K d_{W_{G_n}}(x)\mathbf{1}\{d_{W_{G_n}}(x)\le M\}\,\mathrm{d}x = 0. \tag{5.5}$$

This implies, $\limsup_{n\to\infty}\int_0^K d_{W_{G_n}}(x)\mathbf{1}\{d_{W_{G_n}}(x)\le M\}\,\mathrm{d}x = 0$, for all M, since

$$\limsup_{n\to\infty}\int_0^K d_{W_{G_n}}(x)\mathbf{1}\{d_{W_{G_n}}(x)\le M\}\,\mathrm{d}x$$

is nondecreasing in M. This shows (1.12) with $d = 0$.

Next, we will show that $\lim_{n\to\infty}\|W_{G_n}\|_{L_1([0,K]^2)} = 0$, for every $K > 0$ (which implies (1.11) holds with $W = 0$, since $\lim_{n\to\infty}\|W_{G_n}\|_{\square([0,K]^2)} \le \lim_{n\to\infty}\|W_{G_n}\|_{L_1([0,K]^2)} = 0$). To this end, we have

$$\begin{aligned}
&\int_{[0,K]^2} W_{G_n}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \int_{[0,K]^2} W_{G_n}(x,y)\mathbf{1}\{d_{W_{G_n}}(x)\le M\}\,\mathrm{d}x\,\mathrm{d}y + \int_{[0,K]^2}\frac{d_{W_{G_n}}(x)}{M}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_0^K d_{W_{G_n}}(x)\mathbf{1}\{d_{W_{G_n}}(x)\le M\}\,\mathrm{d}x + \frac{K}{M}\int_0^K d_{W_{G_n}}(x)\,\mathrm{d}x.
\end{aligned}$$

On letting $n \to \infty$ followed by $M \to \infty$, the first term above converges to 0 by (5.5), and the second term converges to 0 by (1.6). This implies W_{G_n} converges to 0 on $L_1([0,K]^2)$, verifying condition (1.11) in Theorem 1.1 with $W = 0$.

Finally, we verify condition (a) of Theorem 1.1. Using (3.1) note that for all K large enough there exists an integer $n(K)$, such that if $n > n(K)$, we have $d_v < r_n$, for all $v \in [\lceil Kr_n\rceil + 1, n]$. In particular, for $M > 1$ this implies

$$\begin{aligned}
0 &\le \int_0^\infty\int_0^\infty W_{G_{n,M}}(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_K^\infty\int_K^\infty W_{G_n}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\le 2\int_0^\infty\int_0^K W_{G_n}(x,y)\mathbf{1}\{d_{W_{G_n}}(x)\le M\}\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_0^K d_{W_{G_n}}(x)\mathbf{1}\{d_{W_{G_n}}(x)\le M\}\,\mathrm{d}x,
\end{aligned}$$

which converges, under the double limit, to 0 by (5.5), for every $K \ge 0$ fixed. Hence,

$$\begin{aligned}
\lim_{K\to\infty}\lim_{n\to\infty}\frac{1}{2}\int_K^\infty\int_K^\infty W_{G_n}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \lim_{M\to\infty}\lim_{n\to\infty}\frac{1}{2}\int_0^\infty\int_0^\infty W_{G_{n,M}}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \lim_{M\to\infty}\lim_{n\to\infty}\mathbb{E}T_{n,M} = \lambda,
\end{aligned}$$

verifying condition (a) of Theorem 1.1. This completes the proof of $(b) \Rightarrow (c)$.

To prove $(a) \Rightarrow (b)$, recall the definition of $T_{n,M}$ from (1.17), and use Lemma 3.1 to note that $T_{n,M} \xrightarrow{D} \mathrm{Pois}(\lambda)$, under the double limit as $n \to \infty$ followed by $M \to \infty$. Now, by a diagonalization argument, given any subsequence we can find a further subsequence $\{n_j\}_{j\ge1}$ such that $\mu_{a,M} := \lim_{j\to\infty}\mathbb{E}T^a_{n_j,M}$ converges, for all $a \ge 1$, by uniform integrability, since the moments $\sup_{n\in N}\mathbb{E}T_{n,M}^a \lesssim_{M,a} 1$, are bounded (recall (3.77)). Recall from the proof of Lemma 3.12 that the moments $\{\mu_{a,M}\}_{a\geq 1}$ satisfy the Carleman moment condition. Therefore, along the subsequence, $T_{n_j,M} \to T_M$, in distribution and in moments, for some random variable T_M. Finally note that the random variables $T_{n_j,M}$ are nondecreasing in M, and so the sequence $\{T_M\}_{M\geq 1}$ is stochastically increasing, and converges in distribution to $\text{Pois}(\lambda)$. Then, by the monotone convergence theorem, $\mathbb{E}(T_{n_j,M}^a)$ converges to $\mathbb{E}(\text{Pois}(\lambda)^a)$, for all integers $a \geq 1$, under the double limit. In particular, (b) follows from convergence of the first two moments.

5.3. *Proof of Corollary* 1.5. Define $Y_i := X_i\mathbf{1}\{X_i \leq 1\}$, for $1 \leq i \leq n$, and denote by

$$T_n' = \frac{1}{2}\sum_{1\leq u,v\leq n} a_{uv}(G_n)Y_uY_v.$$

To begin with, note that the event $\{|T_n - T_n'| > 0\}$ is contained in the following event: there exists $(u,v) \in E(G_n)$ such that either $\{X_u \geq 2 \text{ and } X_v \geq 1\}$ or $\{X_u \geq 1 \text{ and } X_v \geq 2\}$. Therefore, by a union bound,

$$\begin{aligned}\mathbb{P}(|T_n - T_n'| > 0) &\leq 2\sum_{(u,v)\in E(G_n)} \mathbb{P}(X_u \geq 2)\mathbb{P}(X_v \geq 1) = 2|E(G_n)|\mathbb{P}(X_1 \geq 2)\mathbb{P}(X_1 \geq 1)\\ &= |E(G_n)|o(p_n^2),\end{aligned}$$

using $\mathbb{P}(X_1 \geq 1) \leq \mathbb{E}(X_1) = O(p_n)$ and $2\mathbb{P}(X_1 \geq 2) \leq \mathbb{E}(X_1) - \mathbb{P}(X_1 = 1) = o(p_n)$ by the assumption that $\lim_{n\to\infty}\frac{1}{p_n}\mathbb{E}X_1 = 1$. Since $|E(G_n)|p_n^2 = O(1)$ by assumption (1.6), it follows that $T_n - T_n' \xrightarrow{P} 0$. Using $T_n' \xrightarrow{D} Q_1 + Q_2 + Q_3$, by Theorem 1.1, the result follows.

APPENDIX A: APPROXIMATION BY BLOCK FUNCTIONS

In this section we show that a L-block approximation of a L_1-integrable function converges to the function in L_1. This result has been used in the proof of Theorem 1.1.

PROPOSITION A.1. *Suppose $f : [0,1]^d \to \mathbb{R}$ is a bounded measurable function. For any integer $L \geq 1$ define the function $f_L : [0,1]^d \to \mathbb{R}$ as,*

$$f_L(x_1, x_2, \dots, x_d) = L^d \prod_{i=1}^d \int_{\frac{\lceil Lx_i\rceil - 1}{L}}^{\frac{\lceil Lx_i\rceil}{L}} f(y_1, y_2, \dots, y_d)\,\mathrm{d}y_1\,\mathrm{d}y_2\cdots\mathrm{d}y_d.$$

Then $\|f_L - f\|_{L_1([0,1]^d)} = \int_{[0,1]^d}|f_L(\boldsymbol{x}) - f(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \to 0$, as $L \to \infty$.

PROOF. Throughout the proof we abbreviate the norm $\|\cdot\|_{L_1([0,1]^d)}$ as $\|\cdot\|_1$. Now, fixing $\varepsilon > 0$, by standard measure theory arguments, there exists a continuous function $g : [0,1]^d \to \mathbb{R}$ such that $\sup_{\boldsymbol{x}\in[0,1]^d}|g(\boldsymbol{x})| \leq \sup_{\boldsymbol{x}\in[0,1]^d}|f(\boldsymbol{x})|$, and $\|f - g\|_1 \leq \varepsilon$. Then using Jensen's inequality, $\|f_L - g_L\|_1 \leq \|f - g\|_1 \leq \varepsilon$. An application of triangle inequality then gives,

$$\|f - f_L\|_1 \leq \|f - g\|_1 + \|g - g_L\|_1 + \|g_L - f_L\|_1 \leq 2\varepsilon + \|g - g_L\|_1,$$

which implies $\limsup_{L\to\infty}\|f - f_L\|_1 \leq 2\varepsilon$, since $\|g_L - g\|_1 \leq \|g_L - g\|_\infty \to 0$, by the continuity of g. This completes the proof since $\varepsilon > 0$ is arbitrary. □

APPENDIX B: STOCHASTIC INTEGRATION WITH RESPECT TO A POISSON PROCESS

Let $\mathcal{X} = [0, \infty)$, $\mathscr{B}(\mathcal{X})$ the Borel sigma-algebra on $\mathcal{X}$, and λ is the Lebesgue measure on $(\mathcal{X}, \mathscr{B}(\mathcal{X}))$. Denote by $L_p(\mathcal{X}^d)$ the set of all Borel measurable functions $f : \mathcal{X}^d \to \mathbb{R}$ such that $\int_{\mathcal{X}^d} |f(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} < \infty$, where $\boldsymbol{x} = (x_1, x_2, \dots, x_d)$ and $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1 \cdots \mathrm{d}x_d$, with respect to the Lebesgue measure on $\mathcal{X}^d$. In this section, we define stochastic integration with respect to a Poisson process, for functions in $L_1(\mathcal{X}^d)$. The theory of multiple stochastic integration for square integrable functions, with respect to a general centered Levy process is well-understood (see [16] and the references therein). However, our applications require integration of functions in L_1 (for example, the function $\Delta(\cdot)$ in Theorem 1.1 is in $L_1(\mathcal{X})$, but not in $L_2(\mathcal{X})$). In this section, we make the necessary modifications to the standard theory, extending stochastic integration with respect to a Poisson process to L_1 functions.

Let $\{N(A), A \in \mathscr{B}(\mathcal{X})\}$ be the homogenous Poisson process of rate 1 (that is, $N(A) \sim \mathrm{Pois}(\lambda(A))$, where λ is the Lebesgue measure on $(\mathcal{X}, \mathscr{B}(\mathcal{X}))$), defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Denote by $\mathcal{E}_d$ the set of all Itô-elementary functions, having the form

$$f(t_1, t_2, \dots, t_d) = \sum_{i_1, i_2, \dots, i_d = 1}^{m} a_{i_1, i_2, \dots, i_d} \mathbf{1}_{A_{i_1} \times \cdots \times A_{i_d}}(t_1, t_2, \dots, t_d), \tag{B.1}$$

where $A_1, A_2, \dots, A_m \in \mathscr{B}(\mathcal{X})$ are pairwise disjoint, and $a_{i_1, i_2, \dots, i_d}$ is zero if two indices are equal. Note that an Itô-elementary function need not necessarily be in $L_1(\mathcal{X}^d)$. We begin by defining multiple Itô integrals for functions in $\mathcal{E}_d \cap L_1(\mathcal{X}^d)$.

DEFINITION B.1 (Multiple Itô integral for elementary functions). The d-dimensional Itô-stochastic integral, with respect to the Poisson process $\{N(A), A \in \mathcal{B}(\mathcal{X})\}$, for the function $f \in \mathcal{E}_d \cap L_1(\mathcal{X}^d)$ in (B.1) is defined as

$$I_d(f) := \int f(x_1, x_2, \dots, x_d) \prod_{a=1}^{d} \mathrm{d}N(x_a) := \sum_{i_1, i_2, \dots, i_d = 1}^{m} a_{i_1, i_2, \dots, i_d} N(A_{i_1}) \times \cdots \times N(A_{i_d}).$$

It is easy to verify that this is well defined, that is, if $f, g \in \mathcal{E}_d \cap L_1(\mathcal{X}^d)$, with $f = g$ almost everywhere Lebesgue, then $I_d(f) \stackrel{\text{a.s.}}{=} I_d(g)$. The multiple Itô integral for elementary functions also satisfies the following two properties:

- (Finiteness) $|I_d(f)| < \infty$ almost surely, for $f \in \mathcal{E}_d \cap L_1(\mathcal{X}^d)$. To see this note that $\mathbb{E}[N(A_{i_1}) \times \cdots \times N(A_{i_d})] = \lambda(A_{i_1}) \times \cdots \times \lambda(A_{i_d})$, where $\lambda(A)$ denotes the one-dimensional Lebesgue measure of the set A, whenever all the indices $i_1, i_2, \dots, i_d$ are distinct. Therefore,

$$\mathbb{E}\big[|I_d(f)|\big] \le \sum_{i_1, i_2, \dots, i_d = 1}^{m} |a_{i_1, i_2, \dots, i_d}| \lambda(A_{i_1}) \times \cdots \times \lambda(A_{i_d}) = \int_{\mathcal{X}^d} \big|f(\boldsymbol{x})\big| \, \mathrm{d}\boldsymbol{x} < \infty. \tag{B.2}$$

- (Linearity) Given two simple functions $f, g \in \mathcal{E}_d \cap L_1(\mathcal{X}^d)$,

$$I_d(f + g) \stackrel{\text{a.s.}}{=} I_d(f) + I_d(g), \tag{B.3}$$

  which is immediate from definitions.

Now, we proceed to define the multiple Itô integral for general functions in $L_1(\mathcal{X}^d)$. To this end, a straightforward modification of the proof of [18], Theorem 2.1, shows that $\mathcal{E}_d$ is dense in $L_1(\mathcal{X}^d)$. Therefore, given $f \in L_1(\mathcal{X}^d)$, there exists a sequence $\{f_n\}_{n \ge 1}$, with $f_n \in \mathcal{E}_d$, such that $\lim_{n \to \infty} \int_{\mathcal{X}^d} |f_n(\boldsymbol{x}) - f(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} = 0$. (Note that this automatically implies $f_n \in \mathcal{E}_d \cap L_1(\mathcal{X}^d)$, for all n large.)

PROPOSITION B.1. *Consider a sequence* $\{f_n\}_{n\geq 1}$, *with* $f_n \in \mathcal{E}_d$, *such that* $\lim_{n\to\infty} \|f_n - f\|_{L_1(\mathcal{X}^d)} = 0$. *Then there exists a random variable* X *defined on* $(\Omega, \mathcal{F}, \mu)$ *such that* $I_d(f_n) \xrightarrow{L_1} X$. *Moreover, if* $\{g_n\}_{n\geq 1}$, *with* $g_n \in \mathcal{E}_d$, *is another sequence such that* $\lim_{n\to\infty} \|g_n - f\|_{L_1(\mathcal{X}^d)} = 0$, *then the sequence of random variables* $\{I_d(f_n)\}_{n\geq 1}$ *and* $\{I_d(g_n)\}_{n\geq 1}$ *converge to the same limit in* $L_1(\Omega)$.

PROOF. Define the sequence $\{h_n\}_{n\geq 1}$ as follows: For $n \geq 1$,

$$h_{2n-1} := f_n \quad \text{and} \quad h_{2n} := g_n.$$

Note that $\lim_{n\to\infty} \|h_n - f\|_{L_1(\mathcal{X}^d)} = 0$. Therefore, given $\varepsilon > 0$, there exists $N(\varepsilon) < \infty$ such that if $n_1, n_2 \geq N(\varepsilon)$, then $\int_{\mathcal{X}^d} |h_{n_1}(\boldsymbol{x}) - h_{n_2}(\boldsymbol{x})|\, \mathrm{d}\boldsymbol{x} < \varepsilon$. This implies,

$$\begin{aligned}
&\text{(by (B.3))} & \mathbb{E}\big|I_d(h_{n_1}) - I_d(h_{n_2})\big| &= \mathbb{E}\big|I_d(h_{n_1} - h_{n_2})\big| \\
&\text{(by (B.2))} & &\leq \int_{\mathcal{X}^d} \big|h_{n_1}(\boldsymbol{x}) - h_{n_2}(\boldsymbol{x})\big|\, \mathrm{d}\boldsymbol{x} \\
& & &< \varepsilon.
\end{aligned}$$

This shows that $\{I_d(h_n)\}_{n\geq 1}$ is Cauchy in $L_1(\Omega)$, and by the completeness of the space $L_1(\Omega)$, the result follows. □

DEFINITION B.2 (Multiple Itô integral for general L_1-functions). The d-dimensional Itô-stochastic integral for a function $f \in L_1(\mathcal{X}^d)$ (denoted as $I_d(f)$) is defined as the L_1 limit of the sequence $\{I_d(f_n)\}_{n\geq 1}$, where $\{f_n\}_{n\geq 1}$ is a sequence such that $f_n \in \mathcal{E}_d$ with $\lim_{n\to\infty} \|f_n - f\|_{L_1(\mathcal{X}^d)} = 0$.

This is well defined by Proposition B.1. Also, as in the case of elementary functions, $I_d(f)$ satisfies the following properties:

- (Finiteness) For any $f \in L_1(\mathcal{X}^d)$,

$$\mathbb{E}\big|I_d(f)\big| \leq \int_{\mathcal{X}^d} \big|f(\boldsymbol{x})\big|\, \mathrm{d}\boldsymbol{x}. \tag{B.4}$$

  To see this, let $\{f_n\}_{n\geq 1}$ be a sequence of elementary functions such that $\lim_{n\to\infty} \|f_n - f\|_{L_1(\mathcal{X}^d)} = 0$. Then using (B.2),

$$\mathbb{E}\big|I_d(f_n)\big| \leq \int_{\mathcal{X}^d} \big|f_n(\boldsymbol{x})\big|\, \mathrm{d}\boldsymbol{x}.$$

  The desired conclusion then follows on letting $n \to \infty$ on both sides of the above inequality, since $\mathbb{E}|I_d(f_n)| \to \mathbb{E}|I_d(f)|$, by Definition B.2.
- (Linearity) For any two functions f, g in $L_1(\mathcal{X}^d)$, $I_d(f+g) \overset{\text{a.s.}}{=} I_d(f) + I_d(g)$, which is immediate from (B.3) and Definition B.2.

The following proposition shows the convergence of stochastic integrals for converging sequence of functions:

PROPOSITION B.2. *Consider a sequence* $\{f_n\}_{n\geq 1}$ *such that* $\lim_{n\to\infty} \|f_n - f\|_{L_1(\mathcal{X}^d)} = 0$. *Then* $I_d(f_n) \xrightarrow{L_1} I_d(f)$ *in* $(\Omega, \mathcal{F}, \mu)$.

PROOF. Note that

$$\mathbb{E}\big|I_d(f_n) - I_d(f)\big| = \mathbb{E}\big|I_d(f_n - f)\big| \leq \int \big|f_n(\boldsymbol{x}) - f(\boldsymbol{x})\big|\, \mathrm{d}\boldsymbol{x},$$

where the first step uses linearity of stochastic integrals, and the second step uses (B.4). Taking limit as $n \to \infty$ on both sides, the result follows. $\square$

We conclude by computing the two-dimensional Itô stochastic integral of the block function (2.1).

EXAMPLE 8. Fix $\kappa > 0$ and consider the B-block function $f : [0, \kappa]^2 \to [0, 1]$ as defined in (2.1). Let $L \geq 1$ and define

$$f^{(L)}(x, y) = \sum_{1 \leq a \neq b \leq \lceil \kappa L \rceil} r_f^{(L)}(a, b) \mathbf{1}\left\{x \in \left[\frac{a-1}{L}, \frac{a}{L}\right]\right\} \mathbf{1}\left\{y \in \left[\frac{b-1}{L}, \frac{b}{L}\right]\right\},$$

where

$$r_f^{(L)}(a, b) := L^2 \int_{\frac{a-1}{L}}^{\frac{a}{L}} \int_{\frac{b-1}{L}}^{\frac{b}{L}} f(u, v) \,\mathrm{d}u \,\mathrm{d}v. \tag{B.5}$$

Note that the sum is over $a \neq b$, that is, $f^{(L)}(x, y) = 0$ when $x, y \in [\frac{a-1}{L}, \frac{a}{L}]$, for some $1 \leq a \leq L$. Therefore, this is the L-step piecewise constant approximation of f, with zeros on the diagonal blocks. By taking L large enough, it follows that $f(x, y) = r_f^{(L)}(a, a) \lesssim 1$, for $x, y \in [\frac{a-1}{L}, \frac{a}{L}]$, which means

$$\sum_{a=1}^{L} \int_{\frac{a-1}{L}}^{\frac{a}{L}} \int_{\frac{a-1}{L}}^{\frac{a}{L}} f(x, y) \,\mathrm{d}x \,\mathrm{d}y \lesssim \frac{1}{L} \to 0.$$

Then by Proposition A.1, $\lim_{L\to\infty} \|f - f^{(L)}\|_{L_1([0,\kappa]^2)} = 0$, which means

$$I_2(f^{(L)}) \xrightarrow{L_1} I_2(f),$$

by Proposition B.2. Now, let $\{N(t) : 0 \leq t \leq \kappa\}$ be a Poisson process of rate 1, and $\partial N(a) := N(\frac{a}{L}) - N(\frac{a-1}{L}) \sim \mathrm{Pois}(1/L)$. Then taking L large enough and Definition B.1,

$$\begin{aligned} I_2(f^{(L)}) &= \sum_{j=1}^{B} b_{jj} \sum_{\lceil c_{j-1}L \rceil \leq a \neq b \leq \lceil c_j L \rceil} \partial N(a) \partial N(b) \\ &\quad + 2 \sum_{1 \leq j < j' \leq B} b_{jj'} \sum_{\substack{\lceil c_{j-1}L \rceil \leq a \leq \lceil c_j L \rceil \\ \lceil c'_{j-1}L \rceil \leq b \leq \lceil c'_j L \rceil}} \partial N(a) \partial N(b) + o_{L_1}(1) \\ &\xrightarrow{L_1} 2 \sum_{j=1}^{B} b_{jj} \binom{N_j}{2} + 2 \sum_{1 \leq j < j' \leq B} b_{jj'} N_j N_{j'}, \\ &= I_2(f), \end{aligned} \tag{B.6}$$

where the $o_{L_1}(1)$-term in the second step goes to zero in L_1 and $\{N_1, N_2, \ldots, N_B\}$ are independent with $N_j \sim \mathrm{Pois}(c_j - c_{j-1})$.

Acknowledgments. The authors thank Shirshendu Ganguly for many illuminating discussions and Jordan Stoyanov for bringing to our attention the reference [19]. The authors also thank the Associate Editor and the anonymous referees for their insightful comments, which greatly improved the presentation of the paper.

Funding. The research of Sumit Mukherjee was partially supported by NSF Grant DMS-1712037.

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