EXPLOSIONS AND ARBITRAGE

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Abstract

The Feller and Khas'minskii tests provide conditions, under which a given diffusion process $X(\cdot)$ in a domain D can have explosions. If such explosions happen with positive probability, what is the distribution $F(t,x) = \mathbb{P}_x(S \leq t)\,,\ 0 \leq t < \infty$ of the explosion time $S = \inf\{t \geq 0 : X(t) \notin D\}$, as a function of the starting position $X(0) = x \in D$? We answer this question in some generality for one-dimensional diffusions in the spirit of McKean (1969), provide specific examples for which explicit computations are possible, and characterize the "survival" function $U(t,x) = 1 - F(t,x)$ as the smallest nonnegative solution of a linear parabolic PDE.

A very similar situation develops when one studies the possibility of outperforming the market portfolio in the context of a Markovian model with n assets. There again, the reciprocal $\mathcal{U}(t,x)$ of the highest return that can be achieved relative to the market over a given time horizon $[0,t]$, using non-anticipative investment rules and starting with an initial configuration of asset prices $X(0) = x \in (0,\infty)^n$, is characterized as the smallest nonnegative supersolution of an appropriate linear partial differential equation of parabolic type.

The connection with explosions also exists, but is now somewhat subtler: it involves an auxiliary probability measure $\mathbb{Q}$, the so-called Föllmer measure, with respect to which the "real" probability measure $\mathbb{P}$ is absolutely continuous and under which the relative market weights become martingales. When outperformance of the market portfolio is possible, these two measures are not equivalent; strict local martingales enter the picture.

In the presence of uncertainty about the local characterisitcs of this Markovian model, we enter the realm of Lyons (1995), and $\mathcal{U}(t,x)$ becomes the smallest nonnegative supersolution of a fully nonlinear partial differential equation of parabolic (HJB-Pucci) type.

Date: July 3, 2013.

References

[1] Fernholz, D. & Karatzas, I. (2010) On optimal arbitrage. *Annals of Applied Probability* **20**, 1179-1204.

[2] Fernholz, D. & Karatzas, I. (2011) Optimal arbitrage under model uncertainty. *Annals of Applied Probability* **21**, 2191-2225.

[3] Karatzas, I. & Ruf, J. (2013) On the distribution of the time-to-explosion for diffusions. *Preprint.*

[4] McKean, H.P. Jr. (1969) *Stochastic Integrals.* Academic Press, New York.

[5] Lyons, T.J. (1995) Uncertain volatility and the risk-free synthesis of securities. *Appl. Math. Finance* **2**, 117-133.

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