Preventing Portfolio Losses by Hedging Maximum Drawdown

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Abstract

In this article, we study the concept of maximum drawdown and its relevance to the prevention of portfolio losses. Maximum drawdown is defined as the largest market drop during a given time interval. We show that maximum drawdown can serve as an additional tool for portfolio managers on top of already existing contracts, such as put or lookback options.

1 Introduction

Market drops are traditionally protected by buying put or lookback options. The payoff of a put option depends on the difference of the strike (a fixed constant benchmark), and the current level of the stock. Thus it has only a limited ability to protect its buyer in the situation that a market drop happens after a market rally, such as during a market bubble. Puts lose their value as the underlying asset increases, and thus it is desirable to buy them when the market reaches its maximum. This leads to a question of optimal market timing for which there is no simple answer.

A natural extension of the portfolio protection is to use a relative rather than an absolute benchmark in measuring the size of the market drop. This is a feature of lookback type contracts which protect against a drop from the running maximum of the asset price. The size of the drop at any given time is also known as the current *drawdown*. However, lookback options may also lose value in from the situation where the market drops, and the drop is followed by a rally. When the market experiences a crash, the lookback option may expire close to worthless if the final asset value is near its running maximum. We show in the following analysis that this situation happens quite often, thus making protection against portfolio loss by lookback options less appealing.

Since both puts and lookbacks may not protect the buyer from a market drop directly, we propose to study the use of *maximum drawdown* for portfolio protection. Maximum drawdown measures the worst loss (largest drawdown) of the market in a given time interval. It also measures the loss associated with the worst possible market timing, meaning buying the asset at its local maximum and selling it at a subsequent local minimum such that this loss is the worst in the given time interval. Maximum drawdown is gaining some popularity among hedge funds, since it can be used as a performance measure of the manager of the portfolio. One can view maximum drawdown as a contingent claim, and price it and hedge it accordingly.

The drawdown has been extensively studied in recent literature. Portfolio optimization using constraints on the drawdown has been considered in Chekhlov, Uryasev and Zabarkin (2005). Harmantzis and Miao (2005) considered the impact of heavy tail returns on the maximum drawdown risk measure. Analytical results linking the maximum drawdown to the mean return appeared in the paper of Magdon-Ismail and Atiya (2004). In a related paper, Magdon-Ismail et. al. (2004) determined the distribution of the maximum drawdown of Brownian motion.

The maximum drawdown option described in this article is a novel concept, although some existing financial contracts have embedded features resembling insurance on a market crash. For instance, equity default swaps (EDS) are triggered by significant drops in the asset value. As for the pricing of EDS, see Albanese and Chen (2005). The list of other possible contracts which depend on the maximum (absolute) drawdown is given in Vecer (2006).

Previous literature on market crises is mostly limited to empirical research as opposed to creating active trading strategies which could hedge out such events. A review of the existing techniques for analysis and potential prediction of such events is given in Sornette (2004). Our research provides additional tools for managing adverse market moves.

2 Maximum Drawdown versus Puts and Lookbacks

Mathematically, the maximum drawdown MDD_T^δ of an asset price S_t is defined as the largest drop of the asset price from its maximum on a given time interval $[T-\delta, T]$. We can write the maximum drawdown MDD_T^δ as

$$MDD_T^\delta = \max_{T-\delta \le t \le T} (M_t - S_t), \tag{1}$$

where $M_t = \max_{T-\delta \le s \le t} S_s$ is the running maximum of the stock. Closely related is the concept of the maximum relative drawdown D_T^δ, defined as

$$D_T^\delta = \sup_{T-\delta \le t \le T} \left(\frac{M_t - S_t}{M_t} \right), \tag{2}$$

the largest percentage drop on a given time interval $[T-\delta, T]$. In particular, notice that the maximum drawdown of the Brownian motion is the maximum relative drawdown of the geometric Brownian motion.

The concepts of the absolute and the relative drawdown are illustrated in Figures 1 and 2. Figure 1 is the S&P500 index for period 01/1970 − 12/2005. Figure 2 is the corresponding maximum drawdown and the maximum relative drawdown for a 3 month moving time interval. Notice that the plot of the maximum (relative) drawdown peaks during the periods of market crises and is low in stable periods. Thus it can serve as an excellent indicator of market stability. The graphs also resemble the Manhattan skyline – it is a mixture of relatively large and relatively small drops.

One of the prime objectives of a portfolio manager is to avoid losses. If the market exhibits a large maximum drawdown, it is not optimal to have a long position in such a market. Another possible solution is to buy protection in terms of an option. As mentioned, a popular solution is to buy either a put option or a lookback option. One could also consider a maximum drawdown option, but this contract is not yet traded. However, maximum drawdown can be replicated by adopting the corresponding hedging strategy.

Both put options and lookback options provide poor protection when the market first falls and then subsequently recovers. The put option can expire out of the money, and the lookback option can have a small payoff if the final value of the asset is close to its running maximum. Figure 3 shows that this is indeed a quite typical case. The graph on the left is a simulated joint density of the S&P 500 index values and its maximum drawdown (3 months) using starting values from January 30th, 2007 (index value 1420.62, volatility 11.5%). Notice that the final value can end up well above the initial index value, but the maximum drawdown corresponding to these particular scenarios can be well above 50 or 60 points. The graph on the right shows the joint density of the drawdown ($M_t - S_t$, the current drop from the running maximum) and the corresponding maximum drawdown. Notice that the most likely situation here is that the drawdown

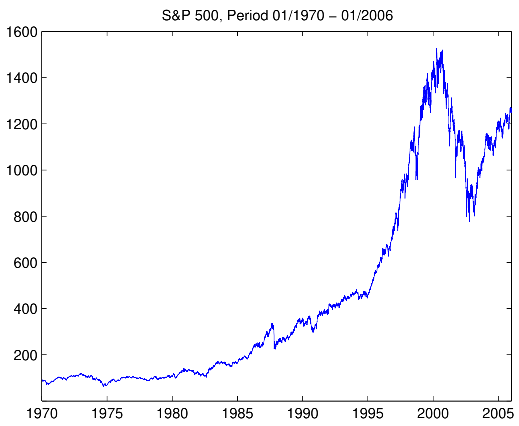


Figure 1: Index S&P 500 from 01/1970 to 12/2005.

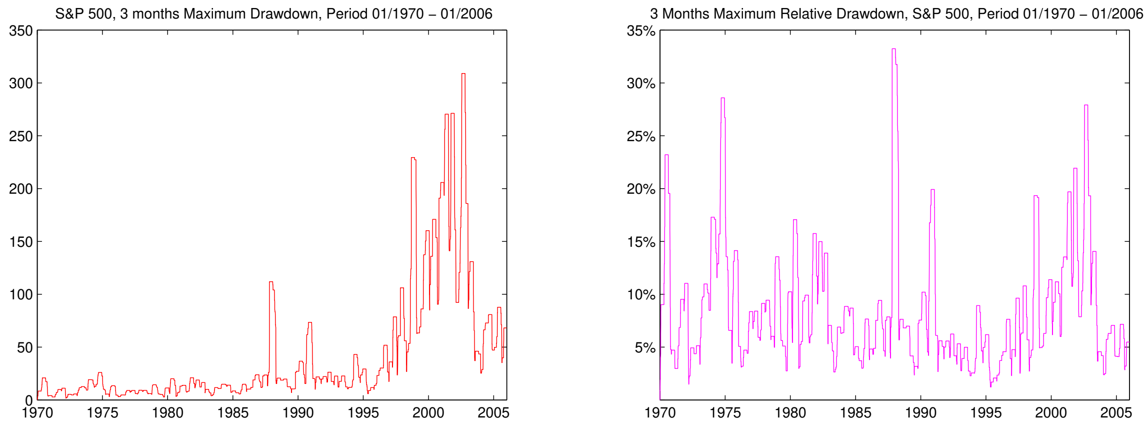


Figure 2: Maximum drawdown for δ=3 months (left) and the corresponding Maximum relative drawdown (right) of the S&P500 index, period from 01/1970 to 12/2005. The graphs clearly indicate the periods of the market crashes.

be at about 10 points, while the maximum drawdown be at about 50 points. In conclusion, having a large maximum drawdown does not mean that either a put option or a lookback option will have the proportional payoff. It is more typical that they will expire with a comparatively small payoff even when the maximum drawdown is relatively large.

3 Pricing and Hedging

Here we apply the elementary techniques of pricing and hedging for the maximum drawdown and compare it to the lookback option. The value $v(t, S_t, M_t, MDD_t)$ of any type of contract depending on the maximum drawdown is given by taking the conditional expectation of the discounted payoff under the risk neutral measure:

$$v(t, S_t, M_t, MDD_t) = \mathbb{E}[e^{-r(T-t)} f(\{MDD_u\}_{u=0}^T) | S_t, M_t, MDD_t].$$

Here, the function f determines the type of payoff defined by the contract (for instance, $f(\{MDD_t\}_{t=0}^T) = MDD_T - K$ for the forward contract, etc.). Notice that the value of this contract depends on three spatial

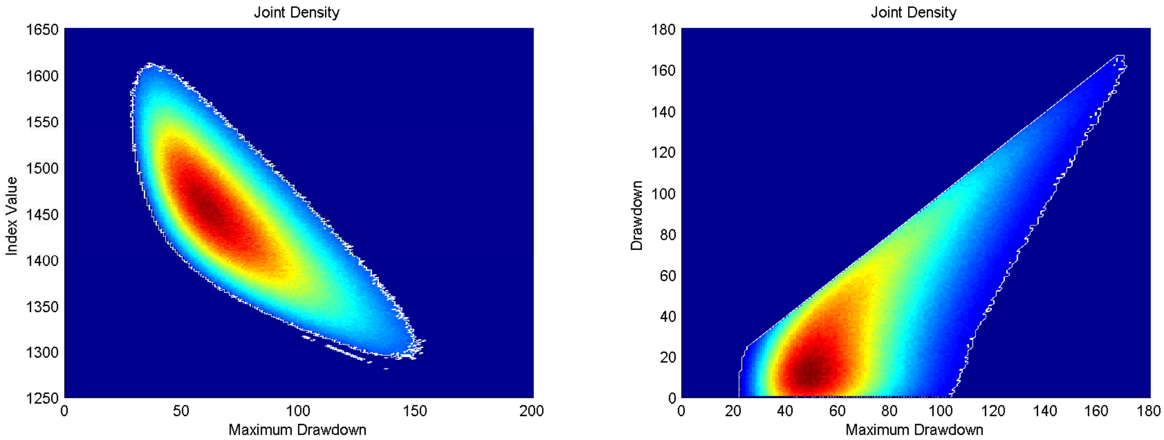


Figure 3: Joint distribution of the S&P 500 and its MDD (left), and joint distribution of the S&P 500 drawdown and MDD (right). Both graphs are on 3 months interval, initial values from January 30th, 2007 (index 1420.62, VIX 11.5%).

variables, namely the current level of the asset price S_t, the current running maximum M_t, and the current maximum drawdown MDD_t.

For the evolution of the underlying asset under the risk neutral measure, we may assume that

$$dS_t = rS_t dt + g(t, S_t) dN_t,$$

for a general martingale N_t (diffusion or jump type process). We assume standard geometric Brownian motion model in the following text. The computation of the conditional expectation is straightforward, one can use Monte-Carlo simulation, or apply partial differential equation techniques. Both methods are discussed in detail in the papers of Vecer (2006), and Pospisil and Vecer (2007).

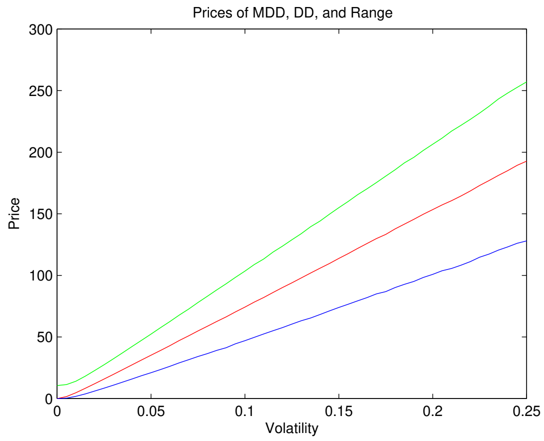


Figure 4: Comparison of the prices of the drawdown (bottom line), the maximum drawdown (middle line), and the range (top line), using initial values of the asset 1420.62, maturity 3 months, interest rate 3%, as a function of volatility.

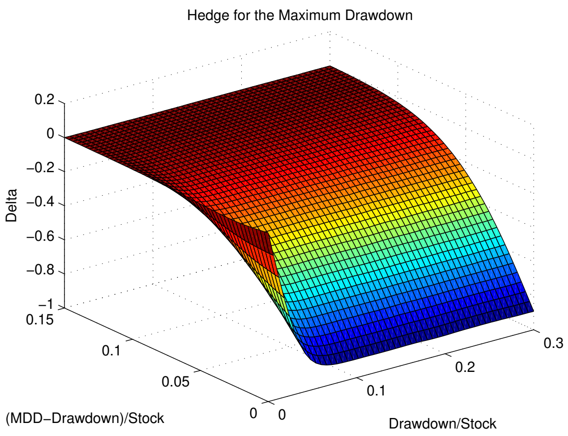
We should mention that the maximum drawdown contract is not a simple combination of the existing lookback option contracts. However it does compare to a range contract, which is the difference of the running maximum M_t and the running minimum m_t in the following way:

$$\text{Range}(t) = M_t - m_t = \max(MDD_t, MDU_t),$$

where MDU_t is the maximum drawup (the largest increase of the price in a given time interval). The range can be replicated by buying a lookback call (payoff $M_t - S_t$) and a lookback put (payoff $S_t - m_t$), but the maximum drawdown coincides with the range only if the maximum drawdown exceeds the maximum drawup. Thus we have the following order relationship:

$$DD_t = M_t - S_t \leq MDD_t \leq \text{Range}(t).$$

Figure 4 shows the price of the lookback call option, the maximum drawdown, and the range contract as a function of volatility. The price of the maximum drawdown is somewhere between the price of the lookback option and the range contract (which is a combination of the lookback call and the lookback put). As a rule of thumb, the price of the MDD contract is approximately 1.5 times the price of the lookback option, and about 0.75 times the price of the range contract.



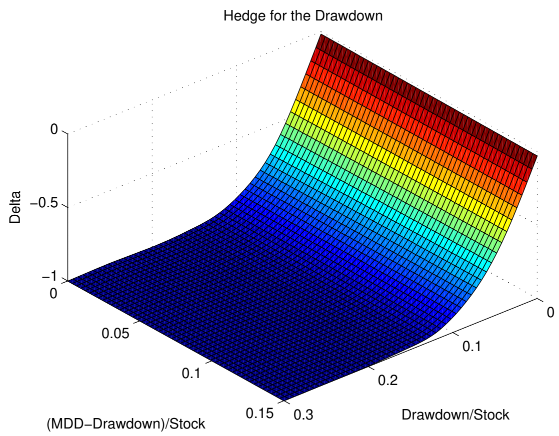


Figure 5: Comparison of the hedging strategies for the maximum drawdown (left), and the lookback option (right).

The hedge of the contract is given by the standard delta hedge

$$\Delta(t, s, m, mdd) = v_s(t, s, m, mdd).$$

An interesting comparison is between the hedge for the maximum drawdown contract, and for the lookback option, see Figure 5. Notice that the hedge for the maximum drawdown shorts the stock only when the current drawdown is close to the maximum drawdown, otherwise the hedging position is insignificant. In contrast, the hedge for the lookback option (or drawdown) shorts each time the current drawdown is large enough without any consideration to the maximum drawdown. This is not surprising since the value of the lookback option does not depend on the maximum drawdown at all. It is also clear from the graph that the hedge for the maximum drawdown requires smaller short positions than the hedge for the lookback option.

4 Case Study: Crash of February 2007

The market drop in late February 2007 serves as an excellent example of what can go wrong with buying a put option or a lookback option. Figure 6 shows the evolution of the S&P 500 index. The market experienced a significant drop on February 27th, followed by a week long decline of the index, but later it recovered and surpassed its original value. Suppose our starting observation point is January 30th, 2007 (an arbitrarily chosen date prior to the market crisis) when the opening value of the index was 1420.62, and the corresponding volatility was 11.5% from VIX index.

A put option with at the money strike and maturity April 30th, 2007 (3 months contract) expired out of the money, thus delivering no protection for its buyer. Similarly, since the final value was close to the running maximum, the lookback option payoff was only 13.05. However, the maximum drawdown (the largest drop) of the index during that period was 85.56! But looking at the joint densities in Figure 3, these are not such unlikely values.

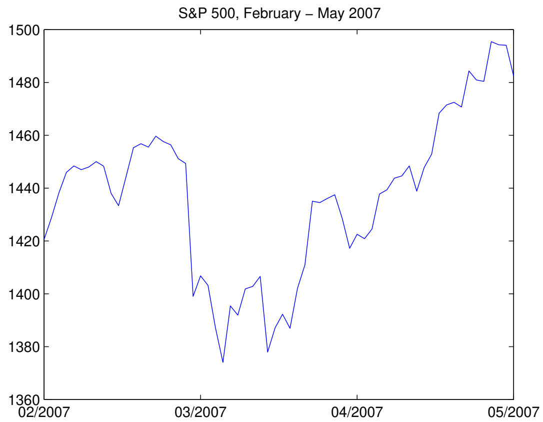


Figure 6: S&P 500 index during February – May 2007, daily closing values.

Consider the following three cases: hedging out the maximum drawdown, hedging out the lookback option, and hedging out the put option struck at the money.

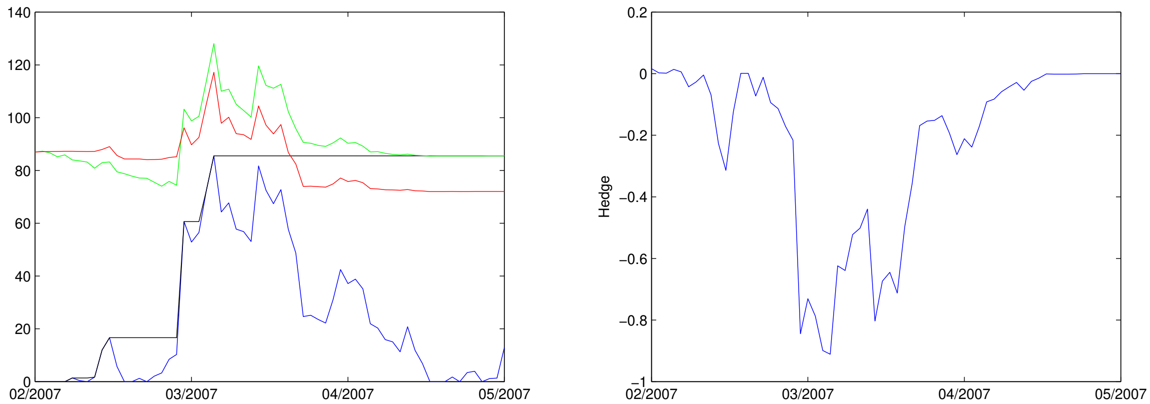
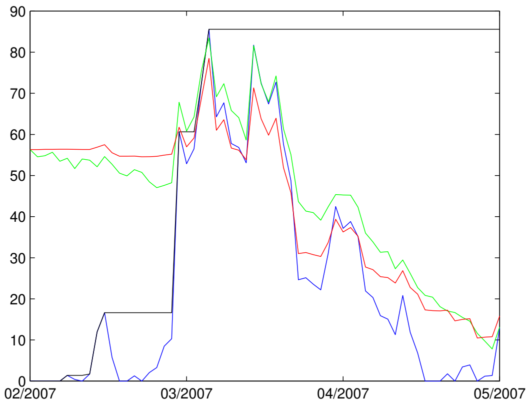


Figure 7: Left: Drawdown (blue), Maximum Drawdown (black), value of the MDD contract (green), replicated portfolio value (red). Right: The corresponding hedge for the MDD contract.

Figure 7 shows the evolution of the price and the replication of a contract on the maximum drawdown. The drawdown (the current drop from the running maximum) is the blue curve, the maximum drawdown is the black curve. When the stock is at its maximum, the drawdown is at zero. The green curve is the theoretical price of the contract on the maximum drawdown. Since this contract is not traded, one can try to replicate it by standard delta hedging. If the market were complete (such as in the geometric Brownian motion model with constant volatility) and we used continuous rebalancing, the replication would be perfect. However, reality may differ from the model, and we also limit ourselves to a simple daily rebalancing which creates some discrepancy between the realized replicating portfolio values and the theoretical value. The replicated portfolio is the red curve and the corresponding hedge is shown on the right graph. The discrepancy between the theoretical value of the contract and our replicated portfolio (slight underhedging) is due to the increased volatility in the post February 27th crash period.

Notice that the expected value of the maximum drawdown for 3 month period was 86.46, close enough to the actual realized value of 85.56. Thus such a contract kept its value even during the market crash in late February 2007. The hedging strategy was to take a significant short position when the drawdown was close to the maximum drawdown (late February and early March 2007), otherwise the hedge was small. During the last month of the contract (April 2007), the hedging position was close to zero. The final value of the replicated portfolio was only at 71.18 due to the increased volatility in the post crash period, resulting in the loss of 15.28 per contract.



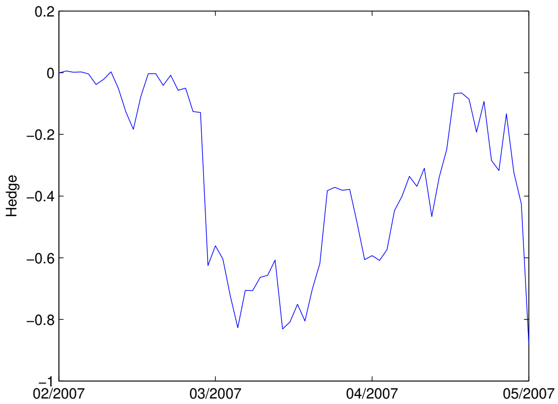


Figure 8: Left: Drawdown (blue), Maximum Drawdown (black), value of the lookback option (green), replicated portfolio value (red). Right: The corresponding hedge for the lookback option.

A natural question here is if the other existing contracts would perform better in this situation. Let us take a look at the corresponding lookback option case in Figure 8. The aim is to price and replicate the drawdown (lookback option), the blue curve in the graph on the left side. The green curve is the theoretical value of the lookback option, the red curve is the replicated portfolio value with the corresponding hedge on the right side of Figure 8. Notice that the replicated portfolio tracks the theoretical value quite closely, but unfortunately the final payoff was only 13.05. The initial value of the lookback option was 54.83, and the terminal value of the replicated portfolio was 14.12, resulting in a total loss of 40.71, almost 3 times more than in the case of the maximum drawdown.

It is interesting to compare the hedge for the maximum drawdown in Figure 7 with the hedge of the lookback option in Figure 8. They are almost identical for the first two months of the contract (February and March 2007), but they differ in the last month (April 2007). The reason is that the market was recovering from previous losses, and the hedge for the maximum drawdown was close to zero when the current drawdown was far from the maximum drawdown. That was the case in April 2007. On the other hand, the hedge for the lookback option was to short each time the drawdown increased, which could happen just because of local daily fluctuations of the price. This resulted in losses for the lookback option contract during the given time period.

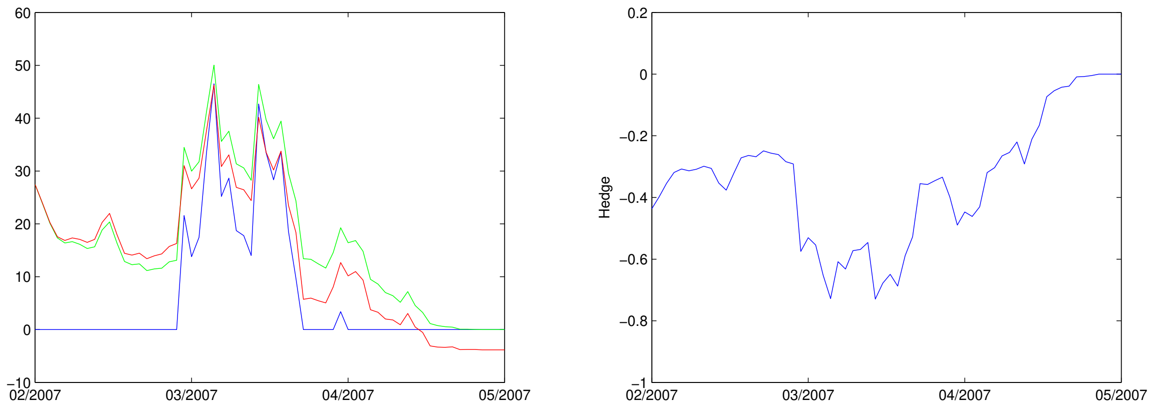


Figure 9: Left: Intrinsic value of the put option $(K - S_t)^+$ (blue), value of the put option (green), replicated portfolio value (red). Right: The corresponding hedge for the put option.

Was the corresponding put option any better? Not quite as seen in Figure 9. The aim is to replicate the blue curve, the intrinsic value of the put option ($(K - S_t)^+$ for $K = 1420.62$). Put options on the index are of a European type for which we can obtain a closed form solution using the Black-Scholes formula. This is the green curve. The replicated portfolio is the red curve which is obtained by using the hedge on the right side of Figure 9. The put option expired at zero, thus its holder would lose the value corresponding to the initial put option price of 27.43. The replicated portfolio finished at -3.82, resulting in a total loss of 31.25. This is more than double the loss associated with the maximum drawdown.

Comparing the hedge of the maximum drawdown (Figure 7) and the put option (Figure 9), there is a remarkable similarity between both contracts in the last month (April 2007). The maximum drawdown behaves almost like the put option in that interval. The significant difference is at the initial stages when the hedge of the put option starts at an already significant short position.

In conclusion, all three contracts (maximum drawdown, lookback option and put option) were able to correctly set the hedge to short prior to the market crash of February 27th. This protection would deliver if sold immediately following this event for all three contracts. This is equivalent to terminating the replication of these contracts at zero cost (the red curves in Figures 7–9). Further losses result if the contracts are kept until expiration. In particular, lookback option is using rather active shorting each time the current drawdown increases. However, this can happen due to local daily fluctuations of the market. In contrast, the hedge for the maximum drawdown relaxes to almost no position if the current drawdown is small compared to the worst drop during the life of the contract.

An argument about the initial expense of these contracts is not entirely relevant here (the maximum drawdown has a higher price than the lookback option, which in turn is typically more expensive than the put option). What matters here is the value of the replicated portfolio, and the replication can be terminated at any time at zero cost (in analogy to futures contracts). The values of these portfolios fluctuate and they are all martingales under the risk neutral measure (effectively a noise with some drift). All three contracts use hedging on a similar scale (from no position or a very small positive position in the case of MDD to the full short position), thus requiring comparable capital involved in the trading.

Ironically, the performance of all these three strategies is suboptimal if compared to a simple long strategy in the market during the studied period. The market ended at 1482.37 on April 30th, 2007, 61.75 points above the starting point of 1420.62. However, there is no general guarantee that the long strategy would work during a market crisis, especially when a crash is preceded by a bubble. In this case, protection by hedging the maximum drawdown is an appropriate strategy.

5 Conclusion

Contingent claims and trading strategies linked to the maximum drawdown introduce new tools for managing adverse market movements. Existing contracts, such as deep out of the money puts, are weakly path dependent and thus have only limited predictive ability of the potential future drawdown. Similarly, lookback type contracts may have a small payoff when the market quickly recovers from a recent loss. When the market is in a bubble, it is reasonable to expect that the prices of drawdown contracts will be significantly higher than when the market is stable, or when it exhibits mean reversion behavior. The prices of contracts linked to the maximum drawdown can serve as an indicator of the risk of future market crises.

References

[1] Albanese, C., O. Chen (2005): "Pricing equity default swaps", *Risk*, Vol. 18, No. 6, 8387.

[2] Chekhlov, A., S. Uryasev, M. Zabarankin (2005): "Drawdown Measure in Portfolio Optimization," *International Journal of Theoretical and Applied Finance,* Vol. 8, No. 1, 13–58.

[3] Harmantzis, F., L. Miao (2005): "On the impact of heavy tails returns to popular risk measures: Evidence from Global Indices", working paper.

[4] Magdon-Ismail, M., A. Atiya (2004): "Maximum drawdown," *Risk*, Vol. 17, No. 10, 99–102.

[5] Magdon-Ismail,M., A. Atiya, A. Pratap, Y. Abu-Mostafa (2004): "On the Maximum Drawdown of a Brownian Motion", *Journal of Applied Probability*, Vol. 41, No. 1, 147-161.

[6] Pospisil, L., J. Vecer (2007): "PDE Methods for Maximum Drawdown", working paper.

[7] Shreve, S. (2004): *Stochastic Calculus for Finance II (Continuous-Time Models)*, *Springer*, New York.

[8] Sornette, D. (2004): *Why Stock Markets Crash*, Princeton Press.

[9] Vecer, J. (2006): "Maximum Drawdown and Directional Trading", *Risk*, Vol. 19, No. 12, 88–92.