GLOBAL TESTING AGAINST SPARSE ALTERNATIVES UNDER ISING MODELS

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APPENDIX – PROOF OF AUXILIARY RESULTS

Proof of Lemma 1. This is a standard application of Stein's Method for concentration inequalities (Chatterjee, 2005). The details are included here for completeness. One begins by noting that

$$\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\left(X_i|X_j, j\neq i\right) = \tanh\left(m_i\left(\mathbf{X}\right)+\mu_i\right), \quad m_i\left(\mathbf{X}\right) := \sum_{j=1}^{n}\mathbf{Q}_{ij}X_j.$$

Now let $\mathbf{X}$ be drawn from (1) and let $\mathbf{X}'$ is drawn by moving one step in the Glauber dynamics, i.e. let I be a random variable which is discrete uniform on $\{1,2,\cdots,n\}$, and replace the I^{th} coordinate of $\mathbf{X}$ by an element drawn from the conditional distribution of the I^{th} coordinate given the rest. It is not difficult to see that $(\mathbf{X},\mathbf{X}')$ is an exchangeable pair of random vectors. Further define an anti-symmetric function $F:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ as $F(\mathbf{x},\mathbf{y})=\sum_{i=1}^n(x_i-y_i)$, which ensures that

$$\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\Big(F(\mathbf{X},\mathbf{X}')|\mathbf{X}\Big) = \frac{1}{n}\sum_{j=1}^{n}X_j - \tanh\left(m_j\left(\mathbf{X}\right)+\mu_j\right) = f_{\boldsymbol{\mu}}(\mathbf{X}).$$

Denoting $\mathbf{X}^i$ to be $\mathbf{X}$ with X_i replaced by $-X_i$, by Taylor's series we have

$$\begin{aligned}
&\tanh(m_j(\mathbf{X}^i)+\mu_j)-\tanh(m_j(\mathbf{X})+\mu_j)\\
=&(m_j(\mathbf{X}^i)-m_j(\mathbf{X}))g'(m_j(\mathbf{X}))+\frac{1}{2}(m_j(\mathbf{X}^i)-m_j(\mathbf{X}))^2g''(\xi_{ij})\\
=&-2\mathbf{Q}_{ij}X_ig'(m_j(\mathbf{X}))+2\mathbf{Q}_{ij}^2g''(\xi_{ij})
\end{aligned}$$

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for some $\{\xi_{ij}\}_{1\le i,j\le n}$, where $g(t) = \tanh(t)$. Thus $f_{\boldsymbol{\mu}}(\mathbf{X}) - f_{\boldsymbol{\mu}}(\mathbf{X}^i)$ can be written as

$$\begin{aligned}
f_{\boldsymbol{\mu}}(\mathbf{X}) - f_{\boldsymbol{\mu}}(\mathbf{X}^i) =& \frac{2X_i}{n} + \frac{1}{n}\sum_{j=1}^{n}\Big\{\tanh\left(m_j\left(\mathbf{X}^i\right)+\mu_j\right) - \tanh\left(m_j\left(\mathbf{X}\right)+\mu_j\right)\Big\} \\
=& \frac{2X_i}{n} - \frac{2X_i}{n}\sum_{j=1}^{n}\mathbf{Q}_{ij}g'(m_j(\mathbf{X})) + \frac{2}{n}\sum_{j=1}^{n}\mathbf{Q}_{ij}^2 g''(\xi_{ij})
\end{aligned}$$

Now setting $p_i(\mathbf{X}) := \mathbb{P}_{\mathbf{Q},\boldsymbol{\mu}}(X_i' = -X_i | X_k, k \neq i)$ we have

$$\begin{aligned}
v(\mathbf{X}) :=& \frac{1}{2}\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\Big(|f_{\boldsymbol{\mu}}(\mathbf{X}) - f_{\boldsymbol{\mu}}(\mathbf{X}')||(X_I - X_I')|\Big|\mathbf{X}\Big) \\
=& \frac{1}{n}\sum_{i=1}^{n}|f_{\boldsymbol{\mu}}(\mathbf{X}) - f_{\boldsymbol{\mu}}(\mathbf{X}^i)|p_i(\mathbf{X}) \\
&\le \frac{2}{n^2}\sum_{i=1}^{n}p_i(\mathbf{X}) - \frac{2}{n^2}\sum_{i,j=1}^{n}|\mathbf{Q}_{ij}p_i(\mathbf{X})g'(m_j(\mathbf{X}))| \\
&+ \frac{2}{n^2}\sum_{i,j=1}^{n}\mathbf{Q}_{ij}^2 g''(\xi_{ij})^2 X_i p_i(\mathbf{X}) \\
\le& \frac{2}{n} + \frac{2}{n^2}\sup_{\mathbf{u},\mathbf{v}\in[0,1]^n}|\mathbf{u}'\mathbf{Q}\mathbf{v}| + \frac{2}{n^2}\sum_{i,j=1}^{n}\mathbf{Q}_{ij}^2,
\end{aligned}$$

where in the last line we use the fact that $\max(|g'(t)|, |g''(t)|) \le 1$. The proof of the Lemma is then completed by an application of Theorem 3.3 of Chatterjee (2007). $\square$

Proof of Lemma 2. Let $\mathbf{Y} := (Y_1, \cdots, Y_n)$ be i.i.d. random variables on $\{-1, 1\}$ with $\mathbb{P}(Y_i = \pm 1) = \frac{1}{2}$, and let $\mathbf{W} := (W_1, \cdots, W_n) \overset{i.i.d.}{\sim} N(0,1)$. Also, for any $t > 0$ let $Z(t\mathbf{Q}, \mu)$ denote the normalizing constant of the p.m.f.

$$\frac{1}{Z(t\mathbf{Q}, \boldsymbol{\mu})}\exp\left(\frac{1}{2}\mathbf{x}^\top t\mathbf{Q}\mathbf{x} + \boldsymbol{\mu}^\top\mathbf{x}\right)$$

Thus we have

$$2^{-n}Z(t\mathbf{Q}, \boldsymbol{\mu}) = \mathbb{E}\mathrm{exp}\Big(\frac{t}{2}\mathbf{Y}^\top\mathbf{Q}\mathbf{Y} + \sum_{i=1}^{n}\mu_i Y_i\Big) \le \mathbb{E}\mathrm{exp}\Big(\frac{t}{2}\mathbf{W}^\top\mathbf{Q}\mathbf{W} + \sum_{i=1}^{n}\mu_i W_i\Big),$$

where we use the fact that $\mathbb{E}Y_i^k \le \mathbb{E}W_i^k$ for all positive integers k. Using spectral decomposition write $\mathbf{Q} = \mathbf{P}^\top\mathbf{\Lambda}\mathbf{P}$ and set $\nu := \mathbf{P}\boldsymbol{\mu}, \widetilde{\mathbf{W}} = \mathbf{P}\mathbf{W}$ to note that

$$\mathbb{E}\exp\Big(\frac{t}{2}\mathbf{W}^\top\mathbf{Q}\mathbf{W}+\sum_{i=1}^n\mu_iW_i\Big)=\mathbb{E}\exp\Big(\frac{t}{2}\sum_{i=1}^n\lambda_i\widetilde{W}_i^2+\sum_{i=1}^n\nu_i\widetilde{W}_i\Big)=\prod_{i=1}^n\frac{e^{\frac{\nu_i^2}{2(1-t\lambda_i)}}}{\sqrt{1-t\lambda_i}}.$$

Combining for any $t>1$ we have the bounds

$$2^n\prod_{i=1}^n\cosh(\mu_i)=Z(\mathbf{0},\boldsymbol{\mu})\le Z(\mathbf{Q},\boldsymbol{\mu})\le Z(t\mathbf{Q},\boldsymbol{\mu})\le 2^n\frac{e^{\sum_{i=1}^n\frac{\nu_i^2}{2(1-t\lambda_i)}}}{\prod_{i=1}^n\sqrt{1-t\lambda_i}}, \tag{1}$$

where the lower bound follows from on noting that $\log Z(t\mathbf{Q},\boldsymbol{\mu})$ is monotone non-decreasing in t, using results about exponential families. Thus invoking convexity of the function $t\mapsto\log Z(t\mathbf{Q},\boldsymbol{\mu})$ we have

$$\begin{aligned}&\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\frac{1}{2}\mathbf{X}^\top\mathbf{Q}\mathbf{X}=\frac{\partial\log Z(t\mathbf{Q},\boldsymbol{\mu})}{\partial t}\Big|_{t=1}\\&\le\frac{\log Z(t\mathbf{Q},\boldsymbol{\mu})-\log Z(\mathbf{Q},\boldsymbol{\mu})}{t-1}\\&\le\sum_{i=1}^n\Big\{\frac{\nu_i^2}{2(1-t\lambda_i)}-\log\cosh(\mu_i)\Big\}-\sum_{i=1}^n\frac{1}{2}\log(1-t\lambda_i),\end{aligned}$$

where we use the bounds obtained in (1). Proceeding to bound the rightmost hand side above, set $t=\frac{1+\rho}{2\rho}>1$ and note that

$$|t\lambda_i|\le\frac{1+\rho}{2}<1.$$

For $x\in\frac{1}{2}[-(1+\rho),(1+\rho)]\subset(-1,1)$ there exists a constant $\gamma_\rho<\infty$ such that

$$\frac{1}{1-x}\le 1+x+2\gamma_\rho x^2,\quad -\log(1-x)\le x+2\gamma_\rho x^2.$$

Also a Taylor's expansion gives

$$-\log\cosh(x)\le-\frac{x^2}{2}+x^4,$$

where we have used the fact that $\|(\log\cosh(x))^{(4)}\|_\infty\le 1$. These, along with the observations that

$$\sum_{i=1}^n\lambda_i=tr(\mathbf{Q})=0,\quad\sum_{i=1}^n\nu_i^2=||\mathbf{P}\boldsymbol{\mu}||^2=||\boldsymbol{\mu}||^2$$

give the bound

$$\begin{aligned}
&\sum_{i=1}^{n}\Big\{\frac{\nu_i^2}{2(1-t\lambda_i)}-\log\cosh(\mu_i)\Big\}-\sum_{i=1}^{n}\frac{1}{2}\log(1-t\lambda_i)\\
\le&\Big\{\frac{1}{2}\sum_{i=1}^{n}\nu_i^2+\frac{t}{2}\sum_{i=1}^{n}\nu_i^2\lambda_i+t^2\gamma_\rho\sum_{i=1}^{n}\nu_i^2\lambda_i^2\Big\}+\Big\{-\frac{1}{2}\sum_{i=1}^{n}\mu_i^2+\sum_{i=1}^{n}\mu_i^4\Big\}+\gamma_\rho t^2\sum_{i=1}^{n}\lambda_i^2\\
=&\frac{t}{2}\boldsymbol{\mu}^\top\mathbf{Q}\boldsymbol{\mu}+t^2\gamma_\rho\boldsymbol{\mu}^\top\mathbf{Q}^2\boldsymbol{\mu}+\sum_{i=1}^{n}\mu_i^4+\gamma_\rho t^2\sum_{i,j=1}^{n}\mathbf{Q}_{ij}^2\\
\le&\frac{t}{2}C\rho\sqrt{n}+t^2\gamma_\rho C\rho^2\sqrt{n}+C\sqrt{n}+\gamma_\rho t^2D\sqrt{n},
\end{aligned}$$

where $D>0$ is such that $\sum_{i,j=1}^{n}\mathbf{Q}_{ij}^2\le D\sqrt{n}$. This along with (1) gives

$$\Big[\frac{1}{2}C(1+t\rho)+t^2\gamma_\rho C\rho^2+C+\gamma_\rho t^2D\Big]\sqrt{n}\ge\frac{1}{2}\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\mathbf{X}^\top\mathbf{Q}\mathbf{X}=\frac{1}{2}\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\sum_{i=1}^{n}X_im_i(\mathbf{X})$$

But, for some random $(\xi_i, i=1,\dots,n)$

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\sum_{i=1}^{n}X_im_i(\mathbf{X})\\
&=\frac{1}{2}\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\sum_{i=1}^{n}\tanh(m_i(\mathbf{X})+\mu_i)m_i(\mathbf{X})\\
&=\frac{1}{2}\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\sum_{i=1}^{n}\tanh(m_i(\mathbf{X}))m_i(\mathbf{X})+\frac{1}{2}\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\sum_{i=1}^{n}\mu_im_i(\mathbf{X})\operatorname{sech}^2(\xi_i).
\end{aligned}$$

Now,

$$\frac{1}{2}\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\sum_{i=1}^{n}\tanh(m_i(\mathbf{X}))m_i(\mathbf{X})\ge\frac{\eta}{2}\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\sum_{i=1}^{n}m_i(\mathbf{X})^2,$$

where

$$\eta:=\inf_{|x|\le1}\frac{\tanh(x)}{x}>0.$$

The desired conclusion of the lemma follows by noting that

$$\Big|\mathbb{E}_{\mathbf{Q},\boldsymbol{\mu}}\sum_{i=1}^{n}\mu_im_i(\mathbf{X})\operatorname{sech}^2(\xi_i)\Big|\le C\sqrt{n}.$$

□

Proof of Lemma 3. We begin with Part (a). By a simple algebra, the p.m.f. of $\mathbf{X}$ can be written as

$$\mathbb{P}_{\theta,\boldsymbol{\mu}}(\mathbf{X}=\mathbf{x}) \propto \exp\left\{\frac{n\theta}{2}\bar{x}^2+\sum_{i=1}^n x_i\mu_i\right\}.$$

Consequently, the joint density of $(\mathbf{X}, Z_n)$ with respect to the product measure of counting measure on $\{-1,1\}^n$ and Lebesgue measure on $\mathbb{R}$ is proportional to

$$\begin{aligned}&\exp\left\{\frac{n\theta}{2}\bar{x}^2+\sum_{i=1}^n x_i\mu_i-\frac{n\theta}{2}(z-\bar{x})^2\right\}\\ =&\exp\left\{-\frac{n\theta}{2}z^2+\sum_{i=1}^n x_i(\mu_i+z\theta)\right\}.\end{aligned}$$

Part (a) follows from the expression above.

Now consider Part (b). Using the joint density of Part (a), the marginal density of Z_n is proportional to

$$\begin{aligned}&\sum_{\mathbf{x}\in\{-1,1\}^n}\exp\left\{-\frac{n\theta}{2}z^2+\sum_{i=1}^n x_i(\mu_i+z\theta)\right\}\\ =&\exp\left\{-\frac{n\theta}{2}z^2+\sum_{i=1}^n \log\cosh(\mu_i+z\theta)\right\}=e^{-f_{n,\boldsymbol{\mu}}(z)},\end{aligned}$$

thus completing the proof of Part (b).

Finally, consider Part (c). By Part (a) given $Z_n=z$ the random variables $(X_1,\cdots,X_n)$ are independent, with

$$\mathbb{P}_{\theta,\boldsymbol{\mu}}(X_i=1|Z_n=z)=\frac{e^{\mu_i+\theta z}}{e^{\mu_i+\theta z}+e^{-\mu_i-\theta z}},$$

and so

$$\mathbb{E}_{\theta,\boldsymbol{\mu}}(X_i|Z_n=z)=\tanh(\mu_i+\theta z),\quad \mathsf{Var}_{\theta,\boldsymbol{\mu}}(X_i|Z_n=n)=\operatorname{sech}^2(\mu_i+\theta z).$$

Thus for any $\boldsymbol{\mu}\in[0,\infty)^n$ we have

$$\begin{aligned}\mathbb{E}_{\theta,\boldsymbol{\mu}}\Big(\sum_{i=1}^n(X_i-\tanh(\mu_i+\theta Z_n))\Big)^2=&\mathbb{E}_{\theta,\boldsymbol{\mu}}\mathbb{E}_{\theta,\boldsymbol{\mu}}\Big\{\Big(\sum_{i=1}^n(X_i-\tanh(\mu_i+\theta Z_n))\Big)^2\Big|Z_n\Big\}\\ =&\mathbb{E}\sum_{i=1}^n\operatorname{sech}^2(\mu_i+\theta Z_n)\le n.\end{aligned}$$

□

Proof of Lemma 4. We begin with Part (a). Since

$$f''_{n,\boldsymbol{\mu}}(z) = \sum_{i=1}^{n} \tanh^2(z+\mu_i)$$

is strictly positive for all but at most one $z \in \mathbb{R}$, the function $z \mapsto f_{n,\boldsymbol{\mu}}(z)$ is strictly convex with $f_{n,\boldsymbol{\mu}}(\pm\infty) = \infty$, it follows that $z \mapsto f_{n,\boldsymbol{\mu}}(z)$ has a unique minima m_n which is the unique root of the equation $f'_{n,\boldsymbol{\mu}}(z) = 0$. The fact that m_n is positive follows on noting that

$$f'_{n,\boldsymbol{\mu}}(0) = -\sum_{i=1}^{n} \tanh(\mu_i) < 0, \quad f'_{n,\boldsymbol{\mu}}(+\infty) = \infty.$$

Also $f'_n(m_n) = 0$ gives

$$m_n = \frac{1}{n}\sum_{i=1}^{n} \tanh(m_n + \mu_i) \le 1,$$

and so $m_n \in (0,1]$. Finally, $f'_{n,\boldsymbol{\mu}}(m_n) = 0$ can be written as

$$m_n - \tanh(m_n) = \frac{s}{n}\Big[\tanh(m_n + B) - \tanh(m_n)\Big] \ge C\frac{s}{n}\tanh(B),$$

for some $C > 0$, which proves Part (a).

Now consider Part (b). By a Taylor's series expansion around m_n and using the fact that $f''_n(z)$ is strictly increasing on $(0,\infty)$ gives

$$f_n(z) \ge f_n(m_n) + \frac{1}{2}(z-m_n)^2 f''_n(m_n + Kn^{-1/4}) \text{ for all } z \in [m_n + Kn^{-1/4}, \infty)$$
$$f_n(z) \le f_n(m_n) + \frac{1}{2}(z-m_n)^2 f''_n(m_n + Kn^{-1/4}) \text{ for all } z \in [m_n, m_n + Kn^{-1/4}].$$

Setting $b_n := f''_n(m_n + Kn^{-1/4})$ this gives

$$\begin{aligned}
&\mathbb{P}_{\theta,\boldsymbol{\mu}}(Z_n > m_n + Kn^{-1/4})\\
=&\frac{\int_{m_n+Kn^{-1/4}} e^{-f_n(z)}dz}{\int_{\mathbb{R}} e^{-f_n(z)}dz}\\
\le&\frac{\int_{m_n+Kn^{-1/4}}^{\infty} e^{-\frac{b_n}{2}(z-m_n)^2}dz}{\int_{m_n}^{m_n+Kn^{-1/4}} e^{-\frac{b_n}{2}(z-m_n)^2}dz}\\
=&\frac{\mathbb{P}(N(0,1) > Kn^{-1/4}\sqrt{b_n})}{\mathbb{P}(0 < N(0,1) < Kn^{-1/4}\sqrt{b_n})},
\end{aligned}$$

from which the desired conclusion will follow if we can show that $\liminf_{n\to\infty} n^{-1/2}b_n > 0$. But this follows on noting that

$$n^{-1/2}b_n = n^{-1/2}f_n''(m_n + Kn^{-1/4})) \geq \sqrt{n}\tanh^2(Kn^{-1/4}) = K^2\Theta(1).$$

Finally, let us prove Part (c). By a Taylor's series expansion about δm_n and using the fact that $f_n(\cdot)$ is convex with unique global minima at m_n we have

$$f_n(z) \geq f_n(m_n) + (z - \delta m_n)f_n'(\delta m_n), \quad \forall z \in (-\infty, \delta m_n].$$

Also, as before we have

$$f_n(z) \leq f_n(m_n) + \frac{1}{2}(z - m_n)^2 f_n''(m_n), \forall z \in [m_n, 2m_n]$$

Thus with $c_n := f_n''(2m_n)$ for any $\delta > 0$ we have

$$\begin{aligned}\mathbb{P}_{\theta,\boldsymbol{\mu}}(Z_n \leq \delta m_n) =& \frac{\int_{-\infty}^{\delta m_n} e^{-f_n(z)}dz}{\int_{\mathbb{R}} e^{-f_n(z)}dz}\\ \leq& \frac{\int_{-\infty}^{\delta m_n} e^{-(z-\delta m_n)f_n'(\delta m_n)}dz}{\int_{m_n}^{2m_n} e^{-\frac{c_n}{2}(z-m_n)^2}dz}\\ =& \frac{\sqrt{2\pi c_n}}{|f_n'(\delta m_n)|\mathbb{P}(0 < Z < m_n\sqrt{c_n})}. \qquad (2)\end{aligned}$$

To bound the the rightmost hand side of (2), we claim that the following estimates hold:

$$c_n = \Theta(nm_n^2), \qquad (3)$$

$$nm_n^3 = O(|f_n'(\delta m_n)|). \qquad (4)$$

Given these two estimates, we immediately have

$$m_n\sqrt{c_n} = \Theta(m_n^2\sqrt{n}) \geq \Theta(A_n^{2/3}\sqrt{n}) \to \infty, \qquad (5)$$

as $A_n \gg n^{-3/4}$ by assumption. Thus the rightmost hand side of (2) can be bounded by

$$\frac{m_n\sqrt{n}}{nm_n^3} = \frac{1}{m_n^2\sqrt{n}} \to 0,$$

where the last conclusion uses (5). This completes the proof of Part (c).

It thus remains to prove the estimates (3) and (4). To this effect, note that

$$
\begin{aligned}
f_n''(2m_n) =& \sum_{i=1}^{n} \tanh^2(2m_n + \mu_i) \\
\leq& \sum_{i=1}^{n} \Big(\tanh(2m_n) + C_1 \tanh(\mu_i)\Big)^2 \\
\leq& 2n \tanh^2(2m_n) + 2C_1^2 \sum_{i=1}^{n} \tanh^2(\mu_i) \\
\lesssim& nm_n^2 + nA(\mu_n) \lesssim nm_n^2,
\end{aligned}
$$

where the last step uses part (a), and $C_1 < \infty$ is a universal constant. This gives the upper bound in (3). For the lower bound of (3) we have

$$
f_n''(m_n) = \sum_{i=1}^{n} \tanh^2(2m_n + \mu_i) \geq n \tanh^2(2m_n) \gtrsim nm_n^3.
$$

Turning to prove (4) we have

$$
\begin{aligned}
|f_n'(\delta m_n)| =& \sum_{i=1}^{n} \tanh(\delta m_n + \mu_i) - n\delta m_n \\
=& \Big[\sum_{i=1}^{n} \tanh(\delta m_n + \mu_i) - \tanh(\delta m_n)\Big] - n[\delta m_n - \tanh(\delta m_n)] \\
\geq& C_2 nA(\mu_n) - C_3 n\delta^3 m_n^3 \\
\gtrsim& nm_n^3,
\end{aligned}
$$

where δ is chosen small enough, and $C_2 > 0, C_3 < \infty$ are universal constants. This completes the proof of (4), and hence completes the proof of the lemma. □

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