

Exact and asymptotic results on coarse Ricci curvature of graphs



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ABSTRACT

Ricci curvature was proposed by Ollivier in a general framework of metric measure spaces, and it has been studied extensively in the context of graphs in recent years. In this paper we obtain the exact formulas for Ollivier's Ricci-curvature for bipartite graphs and for the graphs with girth at least 5. These are the first formulas for Ricci-curvature that hold for a wide class of graphs, and extend earlier results where the Ricci-curvature for graphs with girth 6 was obtained. We also prove a general lower bound on the Ricci-curvature in terms of the size of the maximum matching in an appropriate subgraph. As a consequence, we characterize the Ricci-flat graphs of girth 5. Moreover, using our general lower bound and the Birkhoff–von Neumann theorem, we give the first necessary and sufficient condition for the structure of Ricci-flat regular graphs of girth 4. Finally, we obtain the asymptotic Ricci-curvature of random bipartite graphs $G(n, n, p)$ and random graphs $G(n, p)$, in various regimes of p .

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1. Introduction

Ricci curvature is a fundamental concept in Riemannian geometry, which provides a way of measuring the degree to which the geometry determined by a given Riemannian metric might differ from that of \mathbb{R}^n . Ricci curvature plays an important role in general relativity, where it is the key term in the Einstein field equations, and in the celebrated Ricci flow equation, where a time-dependent Riemannian metric is deformed in the direction of its negative Ricci curvature. Bakry and Émery [2] attempted to define Ricci-curvature through the heat semigroup on a metric measure space. In the recent years, there have been several work on defining a synthetic Ricci curvature on general metric measure spaces by Sturm [27,28], Lott and Villani [20], and Ohta [21].

In the context of graphs, Chung and Yau [5] developed the notion of Ricci-flat graphs, while proving log-Sobolev inequalities. Later, Lin and Yau [18] generalized the notion of Bakry and Émery to the framework of graphs. Finally, Ollivier [23] introduced a notion of coarse Ricci-curvature that extends to Markov chains on metric spaces. This was used to generalize a series of classical theorems in positive Ricci-curvature, such as spectral gap estimates, concentration of measure or log-Sobolev inequalities [22,24]. Joulin and Ollivier [14] proved nonasymptotic estimates for the rate of convergence of empirical means of Markov chains, together with a Gaussian or exponential control on the deviations of empirical means, under the assumption of positive curvature of the underlying space. This assumption reduces to the well-known contraction under path coupling when the underlying space is a finite graph, which has been used extensively to prove fast mixing of several discrete Markov chains (refer to Chapter 14 of Levin et al. [15] for details on path coupling and its application to fast mixing

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and approximate counting of proper q -colorings of a graph). In Riemannian geometry, both Ollivier’s Ricci curvature and Bakry–Émery curvature-dimension inequality give lower bound estimates for the first eigenvalue of the Laplace operator [1,3,22]. For a detailed exposition on geometric methods concerning eigenvalue estimates in the study of Markov chains refer to [6,8,11] and the references therein.

Recently, Ollivier’s Ricci-curvature has been studied in the context of graphs by Jost and Liu [13], Paeng [25], and Cho and Paeng [4]. Very recently, Loisel and Romon [19] described a method for computing Ollivier’s Ricci-curvature for polyhedral surfaces and discussed the connections with linear programming. Lin et al. [17,16] considered a modified definition of Ricci-curvature of graphs, and proved several analogous results. In this paper we study Ollivier’s Ricci-curvature on graphs using the Markov kernel of the simple random walk on the graph. We obtain exact formulas for Ricci-curvature for bipartite graphs and for the graphs with girth at least 5. Previously, an exact formula was only known for graphs with girth at least 6 [4,13]. Our formulas extend these results and provide exact expressions for the Ricci-curvature for a large class of graphs. Using these results we characterize Ricci-flat graphs of girth at least 5. We also prove several other bounds on the Ricci-curvature, involving different graph parameters. The most interesting among them is the derivation of a general lower bound in terms of the size of the maximum matching in an appropriate neighborhood subgraph. Using this we give a necessary and sufficient condition on the structure of Ricci-flat regular graphs of girth 4. Finally, using results about approximate matching in random graphs, we study the asymptotic behavior of Ricci-curvature of random bipartite graphs $G(n, n, p)$ and random graphs $G(n, p)$, in various regimes of p .

1.1. Ollivier’s Ricci-curvature: definitions and notations

In this section we recall some basic facts about Ollivier’s Ricci curvature on graphs and introduce other relevant definitions and notation.

For two probability measures μ_1, μ_2 on a metric space (X, d) , the *transportation distance* (or the *Wasserstein distance*) between them is defined as

$$W_1(\mu_1, \mu_2) = \inf_{\nu \in M(\mu_1, \mu_2)} \int_{X \times X} d(x, y) d\nu(x, y), \tag{1}$$

where $M(\mu_1, \mu_2)$ is the collection of probability measures on $X \times X$ with marginals μ_1 and μ_2 . Another useful representation of the transportation distance is through the celebrated Kantorovich duality (Theorem 1.14, Villiani [29]), which states that

$$W_1(\mu_1, \mu_2) = \sup_{f, 1\text{-Lip}} \left\{ \int_X f d\mu_1 - \int_X f d\mu_2 \right\}, \tag{2}$$

where the supremum is taken over all functions $f : X \rightarrow \mathbb{R}$ which satisfy $|f(x) - f(y)| \leq d(x, y)$, for all $x, y \in X$.

The transportation distance between probability measure is used to define the Ricci-curvature of metric measure spaces. A metric measure space (X, d, m) is a metric space (X, d) , and a collection of probability measures $m = \{m_x : x \in X\}$ indexed by the points of X . The coarse Ricci curvature of a metric measure space is defined as follows:

Definition 1.1 (Ollivier [23]). On any metric measure space (X, d, m) , for any two distinct points $x, y \in X$, the *coarse Ricci curvature* of (X, d, m) of (x, y) is defined as $\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)}$.

Hereafter, we shall refer to Ollivier’s coarse Ricci curvature simply as Ricci curvature and we shall study it for locally finite graphs. Consider a locally finite weighted simple graph $G = (V(G), E(G))$, where each edge $(x, y) \in E(G)$ is assigned a positive weight $w_{xy} = w_{yx}$. The graph is equipped with the standard shortest path graph distance d_G , that is, for $x, y \in V(G)$, $d_G(x, y)$ is the length of the shortest path in G connecting x and y . The *girth* of G is the length of the shortest cycle in G . For $x \in V(G)$ define the degree $d_x = \sum_{(x,y) \in E(G)} w_{xy}$ and the neighborhood $N_G(x) = \{y \in V(G) : (x, y) \in E(G)\}$. For each $x \in V(G)$ define a probability measure

$$m_x(y) = \begin{cases} \frac{w_{xy}}{d_x}, & \text{if } y \in N_G(x) \\ 0, & \text{otherwise.} \end{cases}$$

Note that these are just the transition probabilities of a weighted random walk on the vertices of G . If $m_G = \{m_x : x \in V(G)\}$, then considering the metric measure space $\mathcal{M}(G) := (V(G), d_G, m_G)$, we can define the Ricci curvature for any edge $(x, y) \in E(G)$ as $\kappa_G(x, y) := 1 - W_1^G(m_x, m_y)$. Applying Eq. (1) for $\mathcal{M}(G)$ we get

$$W_1^G(m_x, m_y) = \inf_{\nu \in \mathcal{A}} \sum_{z_1 \in N_G(x)} \sum_{z_2 \in N_G(y)} \nu(z_1, z_2) d(z_1, z_2), \tag{3}$$

where \mathcal{A} denotes the set of all $d_x \times d_y$ matrices with entries indexed by $N_G(x) \times N_G(y)$ such that $\nu(x', y') \geq 0$, $\sum_{z \in N_G(y)} \nu(x', z) = \frac{w_{xx'}}{d_x}$, and $\sum_{z \in N_G(x)} \nu(z, y') = \frac{w_{yy'}}{d_y}$, for all $x' \in N_G(x)$ and $y' \in N_G(y)$. Intuitively, the Wasserstein distance measures the optimal cost to move one pile of sand to another one with the same mass. For a matrix $\nu \in \mathcal{A}$, $\nu(x', y')$ represents the mass moving from $x' \in N_G(x)$ to $y' \in N_G(y)$. For this reason, the matrix ν is often called the *transfer plan*.

By the Kantorovich duality in Eq. (2) we can also write

$$W_1^G(m_x, m_y) = \sup_{f, 1\text{-Lip}} \left\{ \sum_{z \in N_G(x)} f(z)m_x(z) - \sum_{z \in N_G(y)} f(z)m_y(z) \right\}. \tag{4}$$

Henceforth, we denote by \mathcal{L}_1 the set of all 1-Lipschitz functions on G , that is, the set of all functions $f : V(G) \rightarrow \mathbb{R}$ such that $|f(x) - f(y)| \leq d_G(x, y)$, for $x, y \in V(G)$. For any $x \in V(G)$ and any function $f \in \mathcal{L}_1$, define $E_x(f) = \sum_{z \in N_G(x)} f(z)m_x(z)$. With these notation, Eq. (4) now becomes $W_1^G(m_x, m_y) = \sup_{f \in \mathcal{L}_1} \{E_x(f) - E_y(f)\}$. By triangle inequality [23], if $\kappa_G(x, y) \geq \gamma$ for all neighbors $(x, y) \in E(G)$, then $\kappa_G(x, y) \geq \gamma$, for all $x, y \in V(G)$. Therefore, it is reasonable to consider $\kappa_G(x, y)$ only for neighboring vertices $(x, y) \in E(G)$.

Hereafter, the subscript and superscript G from κ_G , W_1^G , and d_G will be often omitted when the graph is clear from the context. For notational brevity, the main theorems will be stated for unweighted graphs, that is, $w_{xy} = 1$, for $(x, y) \in E(G)$. However, most of these results can be generalized to weighted graphs, as discussed in Section 3.3. Finally, for $a, b \in \mathbb{R}$, define $a_+ := \max\{a, 0\}$, $a \wedge b := \min\{a, b\}$, and $a \vee b := \max\{a, b\}$.

1.2. Prior work on Ricci curvature of graphs

Recently, there has been a series of papers on coarse Ricci-curvature when the metric space is a graph G . Jost and Liu proved the following general bounds:

Theorem 1.1 (Jost and Liu [13]). *For any locally finite unweighted graph G , with $(x, y) \in E(G)$,*

$$\frac{|\Delta_G(x, y)|}{d_x \vee d_y} - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{|\Delta_G(x, y)|}{d_x \wedge d_y}\right)_+ - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{|\Delta_G(x, y)|}{d_x \vee d_y}\right)_+ \leq \kappa(x, y) \leq \frac{|\Delta_G(x, y)|}{d_x \vee d_y}, \tag{5}$$

where $\Delta_G(x, y)$ is the number of triangles supported on (x, y) .

They also show that the above lower bound is tight for trees. In fact, it is clear from their proof that the lower bound is an equality whenever there are no 3, 4, or 5 cycles supported on (x, y) . In particular, the lower bound is tight whenever $g(G) \geq 6$, where $g(G)$ is the girth of the graph G . Recently, Cho and Paeng [4] proved this independently, and also showed that the girth condition and the tree formula, obtained by putting $|\Delta_G(x, y)| = 0$ in the lower bound in (5), are equivalent in the following sense: the tree formula holds for all $(x, y) \in E(G)$, if and only if $g(G) \geq 6$. Cho and Paeng [4] also proved other Ricci-curvature bounds involving girth. In particular, they showed that if $g(G) \geq 5$, then $\kappa(x, y) \leq -1 + \frac{2}{\delta}$, where $\delta := \delta(G)$ is the minimum degree in G , and $(x, y) \in E(G)$. They also obtained interesting lower bounds on the clique number and chromatic number of a graph with the Ricci curvature. Paeng [25] used Ollivier’s Ricci curvature to obtain upper bounds on the diameter and volume of finite graphs. The relation between Ollivier’s Ricci curvature and the first eigenvalue was studied by Bauer et al. [3].

Lin et al. [16] introduced a different notion of Ricci-curvature on graphs by modifying Ollivier’s definition. It is defined as the differential limit of a lazy random walk on the graph, and we shall refer to it as the modified Ricci-curvature to distinguish it from Ollivier’s coarse Ricci-curvature. The modified definition has some properties which are similar to the original definition, however, in several contexts they are very different. With this definition, Lin et al. [16] proved a theorem on the modified Ricci curvature of the Cartesian product of graphs. They established upper bounds on the diameter and the number of vertices for graphs with positive curvatures, and also proved asymptotic properties of the modified Ricci-curvature for random graphs. Recently, Lin et al. [17] characterized the set of all modified Ricci-flat graphs with girth at least 5, where a graph is called modified Ricci-flat whenever it has modified Ricci-curvature zero for every edge in the graph. They showed that if G is a connected modified Ricci-flat graph with girth $g(G) \geq 5$, then G is the infinite path, or a cycle C_n with $n \geq 6$, the dodecahedral graph, the Petersen graph, or the half-dodecahedral graph.

1.3. Summary of our results and organization of the paper

In this paper we obtain exact expressions of Ollivier’s Ricci-curvature for bipartite graphs and for the graphs with girth at least 5. For a bipartite graph G with $(x, y) \in E(G)$, we show that the Ricci-curvature has the form

$$\kappa(x, y) = -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \mathcal{E}_G(x, y)\right)_+.$$

The explicit form of $\mathcal{E}_G(x, y)$ is given in Theorem 3.1, and can be interpreted as a non-negative correction term which improves the lower bound in Theorem 1.1 to an exact equality.

Similarly, when G is a graph with girth greater than 4 and $(x, y) \in E(G)$, the Ricci-curvature has the form

$$\kappa(x, y) = -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \mathcal{F}_G(x, y)\right)_+ \wedge - \left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+.$$

The explicit form of $\mathcal{F}_G(x, y)$ is given in Theorem 3.3, and as before, can be interpreted as a non-negative correction term.

To the best of our knowledge, exact formulas for Ricci-curvature were only known for the graphs with girth at least six [4,13]. Our results extend these to graphs with no odd-cycles and graphs with no cycles smaller than 5. As a consequence, we characterize Ricci-flat graphs of girth at least 5, where a graph G is said to Ricci-flat if $\kappa(x, y) = 0$ for all $(x, y) \in E(G)$. This is in analogue to the result on modified Ricci-curvature of Lin et al. [17] in the context of Ollivier's coarse Ricci-curvature.

In Theorem 5.1 we prove a general lower bound on the Ricci-curvature $\kappa(x, y)$ in terms of the size of the matching among the non-common neighbors of x and y in the graph G . This bound is often tight, especially in regular graphs which have a perfect matching between the non-common neighbors of x and y . As the set of all transportation matrices in d -regular graphs is related to the famous Birkhoff polytope, our lower bound result combined with the celebrated Birkhoff–von Neumann theorem gives a necessary and sufficient condition on the structure of Ricci-flat regular graphs of girth 4, which is detailed in Corollary 5.3.

Finally, we also study the Ricci-curvature of random bipartite graphs $G(n, n, p)$ (Theorem 6.2) and random graphs $G(n, p)$ (Theorem 6.3), in various regimes of p . Using a variant of the Hall's marriage theorem, and the existence of near-perfect matching in random bipartite graphs, we obtain the limiting behavior of the Ricci-curvature in the regimes of p where it has a constant limit in probability. We also show that when $np_n \rightarrow \lambda$, that is, the graph is locally tree-like, the Ricci-curvature converges in distribution to the tree formula of Jost and Liu [13]. These are the first known results for Ollivier's Ricci-curvature for Erdős–Renyi random graphs. The analogous version of these results using the modified Ricci-curvature were obtained by Lin et al. [16]. They showed almost sure convergence to constant limits, but could not capture all the different regimes of p .

In Section 2 we prove several important lemmas which build the foundations for proving the main results. We show that the computation of Ricci-curvature on a graph can be formulated as a totally unimodular linear programming problem, and so it suffices to optimize over integer valued 1-Lipschitz functions. We also prove a crucially important reduction lemma where we identify the exact neighborhood of an edge $(x, y) \in E(G)$ that needs to be considered while computing the Ricci-curvature $\kappa(x, y)$. In Section 7 we summarize our work and mention directions for possible future work.

2. Preliminaries

We begin by proving an extension lemma for Lipschitz functions on graphs. Let $G = (V(G), E(G))$ be a locally finite weighted graph. Let $U \subset V(G)$ be a fixed subset of vertices, and d_G the shortest path metric on G .

Lemma 2.1. Any 1-Lipschitz function $g : U \rightarrow \mathbb{R}$, that is, $|g(a) - g(b)| \leq d_G(a, b)$, for $a, b \in U$, can be extended to a 1-Lipschitz function $\bar{g} : V(G) \rightarrow \mathbb{R}$ on G , that is, $|\bar{g}(a) - \bar{g}(b)| \leq d_G(a, b)$, for $a, b \in V$.

Proof. Define $\bar{g} := g$ on U . Let $z \in V(G) \setminus U$ be any point. Note that result follows by induction if we can construct a function $\bar{g} : U \cup \{z\} \rightarrow \mathbb{R}$ which satisfies $|\bar{g}(a) - \bar{g}(b)| \leq d_G(a, b)$, for $a, b \in U \cup \{z\}$. To this end, let

$$A := \bigcap_{a \in U} [\bar{g}(a) - d_G(a, z), \bar{g}(a) + d_G(a, z)].$$

Observe that if A is empty, then there must exist $a, b \in U$ such that $\bar{g}(a) + d_G(a, z) < \bar{g}(b) - d_G(b, z)$. This implies that

$$g(b) - g(a) > d_G(a, z) + d_G(b, z) \geq d_G(a, b).$$

This contradicts the assumption that g is Lipschitz on U , and proves that A is non-empty. Therefore, we can define $\bar{g}(z) = r$, for some $r \in A$. Moreover, by construction $|\bar{g}(a) - \bar{g}(b)| \leq d_G(a, b) \leq d_G(a, b)$, for any two vertices $a, b \in U \cup \{z\}$. By repeating this constructing inductively for every $z \in V(G) \setminus U$, the result follows. \square

Next, we show that computing the transportation distance is a linear programming problem, with integral extreme points. To prove this we use the following result from linear programming (Theorem 2.2, Chapter 4, Yemelichev et al. [30]): The polyhedron $\mathcal{P} = \{\vec{w} : \vec{b}_1 \leq M\vec{w} \leq \vec{b}_2, \vec{d}_1 \leq \vec{w} \leq \vec{d}_2\}$ has integral extreme points, whenever $\vec{b}_1, \vec{b}_2, \vec{d}_1, \vec{d}_2$ are integral vectors and M is totally unimodular, that is, the determinant of every sub-matrix of M is 0, +1, or -1 .

Lemma 2.2. Let $G = (V(G), E(G))$ be a locally finite weighted graph. For $(x, y) \in E(G)$, there exists $g \in \mathcal{L}_1$ such that $g : V(G) \mapsto \mathbb{Z}$, $g(x) = 0$, and $g = \arg \sup_{f \in \mathcal{L}_1} \{E_x(f) - E_y(f)\}$. Thus, while computing $\kappa(x, y)$ it suffices to optimize over integer valued 1-Lipschitz functions, and consequently $\kappa(x, y)$ is rational.

Proof. For $(x, y) \in E(G)$ and $f \in \mathcal{L}_1$, denote by $T_{xy}(f) = E_x(f) - E_y(f) = \frac{1}{d_x} \sum_{z \in N(x)} f(z)w_{xz} - \frac{1}{d_y} \sum_{z \in N(y)} f(z)w_{yz}$. Note that T_{xy} is location invariant, that is, $T_{xy}(f) = T_{xy}(f + c)$, for any $c \in \mathbb{R}$. Therefore, w.l.o.g. generality we can assume $g(x) = 0$. Thus, computing the transportation distance is equivalent to

$$\max_f T_{xy}(f) \text{ subject to } |f(a) - f(b)| \leq 1 \text{ for } (a, b) \in E(G), \text{ and } f(x) = 0.$$

This is clearly a linear programming problem with $|V(G)| - 1$ variables and $2|E(G)|$ constraints.

The set of all feasible functions of this linear program forms a polytope in $\mathbb{R}^{|V(G)|-1}$ with finitely many extreme points. Consider the digraph $D(G)$ obtained by duplicating every edge of G and orienting one each in both directions. As each of the

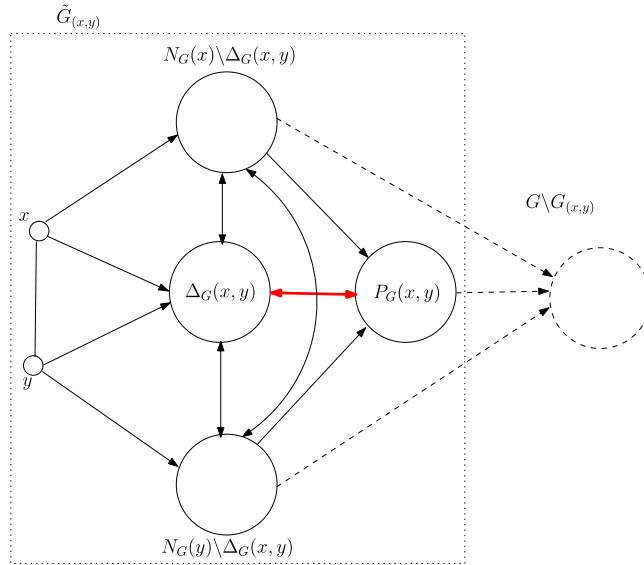


Fig. 1. Core neighborhood of (x, y) : illustration for the proof of the reduction lemma.

constraints of the linear program are of the form $|f(a) - f(b)| \leq 1$ for $(a, b) \in E(G)$, the set of constraints can be written as $M\vec{f} \leq \vec{1}$ and $|\vec{f}| \leq \text{diam}(G) \cdot \vec{1}$, where M is the incidence matrix of the digraph $D(G)$, \vec{f} is the vector of values of f with $f(x) = 0$, and $\text{diam}(G)$ is the diameter of the graph G . As M is totally unimodular (Theorem 13.9, Schrijver [26]), the set of points of this linear program are integral, completing the proof of the lemma. \square

2.1. Reduction lemma: removing large cycles

The transportation distance, and hence the Ricci curvature, of an edge $(x, y) \in E(G)$, is a local property depending only on vertices which are close to x and y . In this section we prove a reduction lemma where we make the above statement precise by exactly identifying the sub-graph of G which contributes to Ricci curvature of an edge $(x, y) \in E(G)$.

Before we proceed to state the lemma, we introduce some notation which will be used throughout the paper. Consider a locally finite weighted graph $G = (V(G), E(G))$. For any two vertices $x, y \in V(G)$, such that $(x, y) \in E(G)$, we associate the following quantities (refer to Fig. 1(a)):

$\Delta_G(x, y)$ This is set of vertices in $V(G)$ which are common neighbors of both x and y , that is, $\Delta_G(x, y) = N_G(x) \cap N_G(y)$. In fact, $|\Delta_G(x, y)|$ is the number of triangles in G supported on (x, y)

$P_G(x, y)$ This is the set of vertices which are at distance 2 from both x and y . That is, $P_G(x, y) = \{v \in V(G) : d_G(x, v) = d_G(y, v) = 2\}$.

Define $V_{(x,y)} := N_G(x) \cup N_G(y) \cup P_G(x, y)$, and denote by H the subgraph of G induced by $V_{(x,y)}$.

Lemma 2.3 (Reduction Lemma). For a locally finite weighted graph $G = (V(G), E(G))$ and an edge $(x, y) \in E(G)$, $\kappa_G(x, y) = \kappa_H(x, y)$. Moreover, for computing $\kappa(x, y)$ it suffices to assume that there are no edges between $\Delta_G(x, y)$ and $P_G(x, y)$.

Proof. To begin with observe that $d_G(a, b) \leq d_H(a, b)$, for any $a, b \in V(H)$. Therefore, any function which is Lipschitz in d_G is also Lipschitz in d_H . Therefore,

$$W_1^H(m_x, m_y) \geq W_1^G(m_x, m_y).$$

To show the other inequality it suffices to show that for any 1-Lipschitz function $f : V(H) \mapsto \mathbb{R}$ with respect to d_H , we can define a function $g : V(G) \mapsto \mathbb{R}$ which is 1-Lipschitz with respect to d_G and agrees with f on $N_G(x, y) := N_G(x) \cup N_G(y)$, as the transportation distance between m_x and m_y only depends on the values of the function at $N_G(x, y)$. To this end, define $g = f$ on $N_G(x, y)$. Observe that if $a, b \in N_G(x, y)$, then by construction of H there is a path from a to b of length $d_G(a, b)$ in H , and so $d_G(a, b) = d_H(a, b)$. Moreover, as f is Lipschitz with respect to d_H , we have $|g(a) - g(b)| = |f(a) - f(b)| \leq d_H(a, b) = d_G(a, b)$. Finally, applying Lemma 2.1 with $U = N_G(x, y)$ proves that g can be extended to a Lipschitz function with respect to d_G on the whole of $V(G)$. This proves that $W_1^H(m_x, m_y) \leq W_1^G(m_x, m_y)$, and the proof of the first part of the lemma is complete.

To show the second part, it suffices to show that $\kappa_H(x, y) = \kappa_{H \setminus e}(x, y)$, for an edge e between $\Delta_G(x, y)$ and $P_G(x, y)$. By the dual definition (3) this means that the optimal transfer matrix \mathcal{A} remains unchanged if we drop the edge e . This is equivalent

to showing that for any $z_1 \in N_G(x)$ and $z_2 \in N_G(y)$ there is a shortest path connecting z_1, z_2 without using e . The following cases may arise:

$d_H(z_1, z_2) = 3$ The shortest path not using e in this case is (z_1, x, y, z_2) .

$d_H(z_1, z_2) = 2$ If $z_1 \in N_G(x) \cap N_G(y)$ then a path of length 2 not containing e is (z_1, y, z_2) . Similarly, for $z_2 \in N_G(x) \cap N_G(y)$ a path of length 2 is (z_1, x, z_2) . Therefore, consider $z_1 \in N_G(x) \setminus N_G(y)$ and $z_2 \in N_G(y) \setminus N_G(x)$. However, it is easy to see that as e is an edge between $\Delta_G(x, y)$ and $P_G(x, y)$, a path of length 2 containing e cannot connect z_1 and z_2 . Therefore, if $d_H(z_1, z_2) = 2$, then there exists a path of length 2 not containing e .

$d_H(z_1, z_2) = 1$ The shortest path in this case is (z_1, z_2) and $e \neq (z_1, z_2)$, as $d_H(z_1, x) = 1, d_H(z_2, y) = 1$. \square

If $\varphi_G(x, y)$ denotes the set of edges between $\Delta_G(x, y)$ and $P_G(x, y)$ (red edges in Fig. 1), we denote the *core neighborhood* of (x, y) in G as the subgraph

$$G_{(x,y)} := (V(H), E(H) \setminus \varphi_G(x, y)).$$

The above lemma shows that it suffices to consider only the core neighborhood subgraph $G_{(x,y)}$ for computing the Ricci-curvature of $(x, y) \in E(G)$, which greatly simplifies computations of $\kappa(x, y)$.

One of the few known exact formulas for Ricci-curvature for graphs is the following result of Jost and Liu [13] for trees:

Theorem 2.1 (Jost and Liu [13]). *For any neighboring vertices x, y of a unweighted tree T ,*

$$\kappa(x, y) = -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+.$$

An immediate consequence of Lemma 2.3 and the above theorem is the following corollary, which generalizes the formula for Ricci curvature of trees to graphs with girth ≥ 6 . This generalization was clear from the proof of Theorem 2.1 in Jost and Lin [13], and was also proved by Cho and Paeng [4].

Corollary 2.2 ([4,13]). *For a locally finite unweighted graph $G = (V(G), E(G))$ and an edge $(x, y) \in E(G)$,*

$$\kappa(x, y) = -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+,$$

if there are no 3, 4, and 5 cycles supported on (x, y) . In particular, the above formula holds whenever the girth of G is at least 6.

3. Bipartite graphs and graphs with girth greater than 4

Jost and Liu [13] obtained bounds on Ricci-curvature involving the number of triangles supported on (x, y) , that is $|\Delta_G(x, y)|$. Their main result is stated in Theorem 1.1, which reduces to the following when there are no triangle supported on (x, y) :

$$-2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+ \leq \kappa(x, y) \leq 0. \tag{6}$$

It is also known that the lower bound is tight for the graphs with girth at least 6 [4,13]. Cho and Paeng [4] also obtained bounds on the Ricci-curvature of graphs with girth at least 5 in terms of the minimum degree of the graph. In this section, using the reduction lemma and by a careful analysis of the structure of the core-neighborhood we improve all existing bounds to obtain exact formulas for the Ricci-curvature in bipartite graphs and graphs with girth at least 5.

3.1. Ricci curvature of bipartite graphs

In this section we shall give an exact formula for $\kappa(x, y)$ whenever the graph G is bipartite, that is, there are no cycles of odd-length in G . We know that $N_G(x)$ and $N_G(y)$ denote the set of neighbors of x and y , respectively. We partition $N_G(x) = N_0(x) \cup N_1(x) \cup \{y\}$, where

$N_1(x) = \{z \in N_G(x) \setminus \{y\} : d_G(z, N_G(y)) = 1\}$, is the set of neighbors of x which are on a 4-cycle supported on (x, y) .

$N_0(x) = N_G(x) \setminus (N_1(x) \cup \{y\})$, is the set of remaining neighbors of x , apart from y .

Similarly, we can define a partition $N_G(y) = N_0(y) \cup N_1(y) \cup \{x\}$. Now, if we assume that G is bipartite, $|\Delta_G(x, y)| = 0$ and $|P_G(x, y)| = 0$. This simplifies the structure of the core neighborhood, and using that we now give the exact formula for $\kappa(x, y)$ in bipartite graphs.

Theorem 3.1. *Let $G = (V(G), E(G))$ be a locally finite unweighted bipartite graph and $(x, y) \in E(G)$. Suppose $R(x, y)$ is the subgraph of $G_{(x,y)}$ induced by $N_1(x) \cup N_1(y)$, and $R_1(x, y), R_2(x, y), \dots, R_q(x, y)$ be the connected components of $R(x, y)$.*

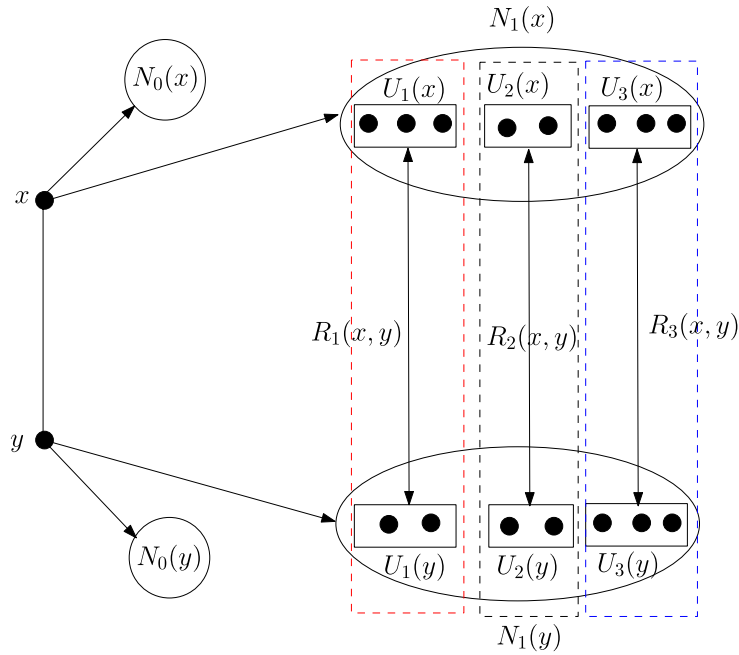


Fig. 2. Ricci curvature of bipartite graphs: structure of the core neighborhood.

If $U_a(x) = V(R_a(x, y)) \cap N_1(x)$ and $U_a(y) = V(R_a(x, y)) \cap N_1(y)$, for $a \in \{1, 2, \dots, q\}$, then

$$\kappa(x, y) = -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{|N_1(y)|}{d_y} + \sum_{a=1}^q \left(\frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right)_+ \right). \tag{7}$$

Proof. Using Lemma 2.3 we can replace G by its core neighborhood $G_{(x,y)}$. As G is bipartite, there are no edges between $N_0(x)$ and $N_1(x)$, and between $N_0(y)$ and $N_1(y)$. Therefore, the structure of the core neighborhood $G_{(x,y)}$ is as depicted in Fig. 2.

For any $Z \subseteq V(G_{(x,y)})$ and any function $f \in \mathcal{L}_1$, denote by $W_f(Z) = \sum_{z \in Z} f(z)$. Therefore, for $(x, y) \in E(G)$,

$$E_y(f) - E_x(f) = \frac{f(x) + W_f(N_0(y)) + W_f(N_1(y))}{d_y} - \frac{f(y) + W_f(N_0(x)) + W_f(N_1(x))}{d_x}. \tag{8}$$

Lemma 2.2 implies that it suffices to maximize $E_y(f) - E_x(f)$ over 1-Lipschitz functions f satisfying $f(x) = 0$, and $f(y) \in \{-1, 0, 1\}$. Therefore, for $i \in \{-1, 0, 1\}$ define

$$\kappa_i(x, y) := 1 - \max_{f \in \mathcal{L}_1, f(y)=i} (E_y(f) - E_x(f)),$$

and observe that $\kappa(x, y) = \kappa_{-1}(x, y) \wedge \kappa_0(x, y) \wedge \kappa_1(x, y)$. Assuming $f(x) = 0$ we consider the following three cases separately.

Case 1:

$f(y) = -1$. This implies that $f(z) \leq 0$ for $z \in N_G(y)$ and $f(z) \geq -1$, for $z \in N_G(x)$. Therefore, from Eq. (8) we get

$$E_y(f) - E_x(f) \leq \frac{1}{d_x} + \frac{|N_0(x)| + |N_1(x)|}{d_x} = 1.$$

Moreover, this bound is attained by the function $g : V(G_{(x,y)}) \mapsto \mathbb{R}$:

$$g(z) = \begin{cases} -1, & \text{if } z \in N_G(x), \\ 0, & \text{otherwise,} \end{cases}$$

which is 1-Lipschitz on the core neighborhood of (x, y) (refer to 2(a)). This implies, $\kappa_{-1}(x, y) = 0$.

Case 2:

$f(y) = 0$. This implies that $f(z) \leq 1$ for $z \in N_G(y)$ and $f(z) \geq -1$, for $z \in N_G(x)$. Therefore, from Eq. (8) we get as before,

$$\begin{aligned} E_y(f) - E_x(f) &\leq \frac{|N_0(y)| + W_f(N_1(y))}{d_y} - \frac{-|N_0(x)| + W_f(N_1(x))}{d_x}, \\ &= \frac{|N_0(y)|}{d_y} + \frac{|N_0(x)|}{d_x} + \frac{W_f(N_1(y))}{d_y} - \frac{W_f(N_1(x))}{d_x}. \\ &= \frac{|N_0(y)|}{d_y} + \frac{|N_0(x)|}{d_x} + \sum_{a=1}^q \left\{ \frac{W_f(U_a(y))}{d_y} - \frac{W_f(U_a(x))}{d_x} \right\}, \end{aligned} \quad (9)$$

where $U_a(x)$ and $U_a(y)$ are as defined in the statement of the theorem, for $a \in \{1, 2, \dots, q\}$. For any 1-Lipschitz function $f : V(G_{(x,y)}) \mapsto \mathbb{Z}$ with $f(x) = f(y) = 0$ denote the restriction of f to $U_a(y) \cup U_a(x) \cup \{x, y\}$ as f_a , for $a \in \{1, 2, \dots, q\}$. This implies that

$$1 - \kappa_0(x, y) \leq \frac{|N_0(y)|}{d_y} + \frac{|N_0(x)|}{d_x} + \sum_{a=1}^q \max_{\substack{f_a \in \mathcal{L}_1, \\ f_a(y)=0}} \left\{ \frac{W_{f_a}(U_a(y))}{d_y} - \frac{W_{f_a}(U_a(x))}{d_x} \right\}. \quad (10)$$

Fix $a \in \{1, 2, \dots, q\}$, and consider a 1-Lipschitz function $f_a : U_a(y) \cup U_a(x) \cup \{x, y\} \mapsto \mathbb{Z}$, with $f_a(x) = f_a(y) = 0$. Define

$$\mathcal{W}(f_a) := \frac{W_{f_a}(U_a(y))}{d_y} - \frac{W_{f_a}(U_a(x))}{d_x}.$$

Now, if there exists $z \in N_1(y)$ such that $f_a(z) = -1$, then for f_a be to 1-Lipschitz $f_a(w) \in \{-1, 0\}$, for all $w \in N_G(z) \cap N_1(x)$. Then the function $f'_a : U_a(y) \cup U_a(x) \cup \{x, y\} \mapsto \mathbb{Z}$, with $f'_a(z) = 0$ and $f'_a(w) = f_a(w)$, for all $w \neq z$, satisfies $\mathcal{W}(f'_a) < \mathcal{W}(f_a)$. Therefore, it suffices to assume that $f_a(z) \in \{0, 1\}$ for all $z \in N_1(y)$, and similarly $f_a(z) \in \{-1, 0\}$ for all $z \in N_1(x)$. This observation and the bipartite structure of $R_a(x, y)$ immediately implies that there are two possibilities:

- (i) $f_a(z) = 0$ for all $z \in U_a(y)$ and $f_a(z) = -1$ for all $z \in U_a(x)$,
- (ii) $f_a(z) = 1$ for all $z \in U_a(y)$ and $f_a(z) = 0$ for all $z \in U_a(x)$.

This implies that

$$\begin{aligned} \max_{\substack{f_a \in \mathcal{L}_1, \\ f_a(y)=0}} \mathcal{W}(f_a) &= \frac{|U_a(y)|}{d_y} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}} + \frac{|U_a(x)|}{d_x} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} \geq \frac{|U_a(y)|}{d_y} \right\}}. \\ &= \left\{ \frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}} + \frac{|U_a(x)|}{d_x}. \end{aligned} \quad (11)$$

Noting that $\sum_{a=1}^q |U_a(x)| = |N_1(x)|$, and substituting Eq. (11) in Eq. (10) we get

$$\begin{aligned} \kappa_0(x, y) &\geq 1 - \frac{|N_0(y)|}{d_y} - \frac{|N_0(x)| + |N_1(x)|}{d_x} - \sum_{a=1}^q \left\{ \frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}}, \\ &= \frac{1}{d_x} - \frac{|N_0(y)|}{d_y} - \sum_{a=1}^q \left\{ \frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}}. \end{aligned} \quad (12)$$

Now, consider the 1-Lipschitz function $g : V(G_{(x,y)}) \mapsto \mathbb{R}$

$$g(z) = \begin{cases} -1, & \text{if } z \in N_0(x); \\ 1, & \text{if } z \in N_0(y); \\ g_a(z), & \text{if } z \in U_a(x) \cup U_a(y); \\ 0, & \text{otherwise;} \end{cases}$$

where for $a \in \{1, 2, \dots, q\}$,

$$g_a(z) := \begin{cases} -1 \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} \geq \frac{|U_a(y)|}{d_y} \right\}}, & \text{if } z \in U_a(x); \\ 1 \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}}, & \text{if } z \in U_a(y). \end{cases}$$

It is easy to see that

$$E_y(g) - E_x(g) = \frac{|N_0(y)|}{d_y} + \frac{|N_0(x)| + |N_1(x)|}{d_x} + \sum_{a=1}^q \left\{ \frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}}.$$

Therefore, g attains the lower bound in Eq. (12) and we have

$$\kappa_0(x, y) = \frac{1}{d_x} - \frac{|N_0(y)|}{d_y} - \sum_{a=1}^q \left\{ \frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}}. \tag{13}$$

Case 3:

$f(y) = 1$. This implies that $f(z) \leq 2$ for $z \in N_G(y)$ and $f(z) \geq -1$, for $z \in N_G(x)$. Therefore, from Eq. (8) we get as before,

$$E_y(f) - E_x(f) \leq \frac{2|N_0(y)|}{d_y} + \frac{|N_0(x)| - 1}{d_x} + \sum_{a=1}^q \left\{ \frac{W_f(U_a(y))}{d_y} - \frac{W_f(U_a(x))}{d_x} \right\}, \tag{14}$$

where $U_a(x)$ and $U_a(y)$ for $a \in \{1, 2, \dots, q\}$ are as defined in the statement of the theorem. As before, for any 1-Lipschitz function $f : V(G_{(x,y)}) \mapsto \mathbb{Z}$ with $f(x) = 0, f(y) = 1$ denote the restriction of f to $U_a(y) \cup U_a(x) \cup \{x, y\}$ as f_a , for $a \in \{1, 2, \dots, q\}$. Define $W(f_a)$ as before. Then from the bipartite structure of $R_a(x, y)$ it follows that $g_a := \arg \max_{f_a \in \mathcal{L}_1, f_a(y)=1} W(f_a)$ must be one of the following:

- (i) $g_a(z) = 0$ for all $z \in U_a(y)$ and $g_a(z) = -1$ for all $z \in U_a(x)$,
- (ii) $g_a(z) = 1$ for all $z \in U_a(y)$ and $g_a(z) = 0$ for all $z \in U_a(x)$,
- (iii) $g_a(z) = 2$ for all $z \in U_a(y)$ and $g_a(z) = 1$ for all $z \in U_a(x)$.

This implies that,

$$\begin{aligned} \max_{\substack{f_a \in \mathcal{L}_1, \\ f_a(y)=0}} W(f_a) &= \max \left\{ \frac{|U_a(x)|}{d_x}, \frac{|U_a(y)|}{d_y}, \frac{2|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\}, \\ &= \left\{ \frac{2|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}} + \frac{|U_a(x)|}{d_x} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} \geq \frac{|U_a(y)|}{d_y} \right\}}, \\ &= 2 \left\{ \frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}} + \frac{|U_a(x)|}{d_x}. \end{aligned} \tag{15}$$

Noting that $\sum_{a=1}^q |U_a(x)| = |N_1(x)|$, and substituting Eq. (15) in Eq. (14) we get

$$\kappa_1(x, y) \geq \frac{2}{d_x} - \frac{2|N_0(y)|}{d_y} - 2 \sum_{a=1}^q \left\{ \frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}}. \tag{16}$$

Now, consider the 1-Lipschitz function $g : V(G_{(x,y)}) \mapsto \mathbb{R}$

$$g(z) = \begin{cases} -1, & \text{if } z \in N_0(x); \\ 1, & \text{if } z \in N_0(y); \\ g_a(z), & \text{if } z \in U_a(x) \cup U_a(y); \\ 0, & \text{otherwise;} \end{cases}$$

where for $a \in \{1, 2, \dots, q\}$,

$$g_a(z) := \begin{cases} -1 \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} \geq \frac{|U_a(y)|}{d_y} \right\}} + 1 \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}}, & \text{if } z \in U_a(x); \\ 2 \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}}, & \text{if } z \in U_a(y). \end{cases}$$

It is easy to see that g attains the lower bound in Eq. (16) and we have

$$\kappa_1(x, y) = \frac{2}{d_x} - \frac{2|N_0(y)|}{d_y} - 2 \sum_{a=1}^q \left\{ \frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} < \frac{|U_a(y)|}{d_y} \right\}}. \tag{17}$$

Note that $\kappa_1(x, y) = 2\kappa_0(x, y)$ and $\kappa_{-1}(x, y) = 0$. Therefore, $\kappa(x, y) = \kappa_{-1}(x, y) \wedge \kappa_0(x, y) \wedge \kappa_1(x, y) = \kappa_1(x, y) \wedge 0$, and the proof completes. \square

The formula in the above theorem is exact and, as a result, it is quite complicated and heavily depends on the structure of the graph.

Remark 3.1. It can be easily verified that (7) is symmetric in x and y . To see this, note by (7)

$$\begin{aligned}\kappa(y, x) &= -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{|N_1(x)|}{d_x} + \sum_{a=1}^q \left\{ \frac{|U_a(x)|}{d_x} - \frac{|U_a(y)|}{d_y} \right\} \cdot \mathbf{1}_{\left\{ \frac{|U_a(x)|}{d_x} \geq \frac{|U_a(y)|}{d_y} \right\}} \right)_+ \\ &= -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{|N_1(x)|}{d_x} + \sum_{a=1}^q \left\{ \frac{|U_a(x)|}{d_x} - \frac{|U_a(y)|}{d_y} \right\} + \sum_{a=1}^q \left(\frac{|U_a(y)|}{d_y} - \frac{|U_a(x)|}{d_x} \right)_+ \right)_+ \\ &= \kappa(x, y),\end{aligned}$$

since $\sum_{a=1}^q \frac{|U_a(x)|}{d_x} = \frac{|N_1(x)|}{d_x}$ and $\sum_{a=1}^q \frac{|U_a(y)|}{d_y} = \frac{|N_1(y)|}{d_y}$.

In the following corollary we derive a cleaner upper bound on $\kappa(x, y)$ for bipartite graphs, which is tight whenever the graph G is connected.

Corollary 3.2. Let $G = (V(G), E(G))$ be a locally finite bipartite graph and $(x, y) \in E(G)$. Then

$$\kappa(x, y) \leq -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \left\{ \frac{|N_1(x)|}{d_x} \wedge \frac{|N_1(y)|}{d_y} \right\} \right)_+,$$

and equality holds whenever $R(x, y)$ (defined in Theorem 3.1) is connected. In particular, for the complete bipartite graph $K_{p,q}$ ($p \leq q$), $\kappa(x, y) = 0$, for any $(x, y) \in E(K_{p,q})$.

Proof. The proof of the first part follows from Eq. (7) by dropping the last non-negative term. The second part follows from (7) by direct substitution, from which the result on $K_{p,q}$ follows. \square

3.2. Ricci curvature of graphs with girth greater than 4

In this section we shall give an exact formula for $\kappa(x, y)$ for graphs G with girth greater than 4. As before, partition $N_G(x) = N_0(x) \cup N_2(x) \cup \{y\}$, where

$$N_2(x) = \{z \in N_G(x) \setminus \{y\} : d_G(z, N_G(y)) = 2\}.$$

$N_0(x) = N_G(x) \setminus (N_2(x) \cup \{y\})$, is the set of remaining neighbors of x , apart from y .

Similarly, we can define a partition $N_G(y) = N_0(y) \cup N_2(y) \cup \{x\}$. Using these definitions we now give the exact formula for $\kappa(x, y)$ for graphs G with girth greater than 4.

Theorem 3.3. Let $G = (V(G), E(G))$ be a locally finite unweighted graph with girth $g(G) \geq 5$ and $(x, y) \in E(G)$. Suppose $Q(x, y)$ is the subgraph of $G_{(x,y)}$ induced by $N_2(x) \cup N_2(y) \cup P_G(x, y)$, and $Q_1(x, y), Q_2(x, y), \dots, Q_q(x, y)$ be the connected components of $Q(x, y)$. If $L_a(x) = V(Q_a(x, y)) \cap N_2(x)$ and $L_a(y) = V(Q_a(x, y)) \cap N_2(y)$, for $a \in \{1, 2, \dots, q\}$, then $\kappa(x, y) = \kappa_0(x, y) \wedge \kappa_1(x, y)$, where

$$\kappa_0(x, y) = - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+ \tag{18}$$

and

$$\kappa_1(x, y) = -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{|N_2(x)|}{2d_x} + \frac{1}{2} \sum_{a=1}^q \left(\frac{|L_a(x)|}{d_x} - \frac{|L_a(y)|}{d_y} \right)_+ \right)_+ \tag{19}$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. Using Lemma 2.3 we can replace G by its core neighborhood $G_{(x,y)}$, which is depicted in Fig. 3. For any $Z \subseteq V(G_{(x,y)})$ and any function $f \in \mathcal{L}_1$, denote by $W_f(Z) = \sum_{z \in Z} f(z)$. Therefore, for $(x, y) \in E(G)$,

$$E_y(f) - E_x(f) = \frac{f(x) + W_f(N_0(y)) + W_f(N_2(y))}{d_y} - \frac{f(y) + W_f(N_0(x)) + W_f(N_2(x))}{d_x}. \tag{20}$$

As before, for $i \in \{-1, 0, 1\}$ define

$$\kappa_i(x, y) := 1 - \max_{f \in \mathcal{L}_1, f(y)=i} (E_y(f) - E_x(f)),$$

and observe that $\kappa(x, y) = \kappa_{-1}(x, y) \wedge \kappa_0(x, y) \wedge \kappa_1(x, y)$. Assuming $f(x) = 0$ we consider the following three cases separately.

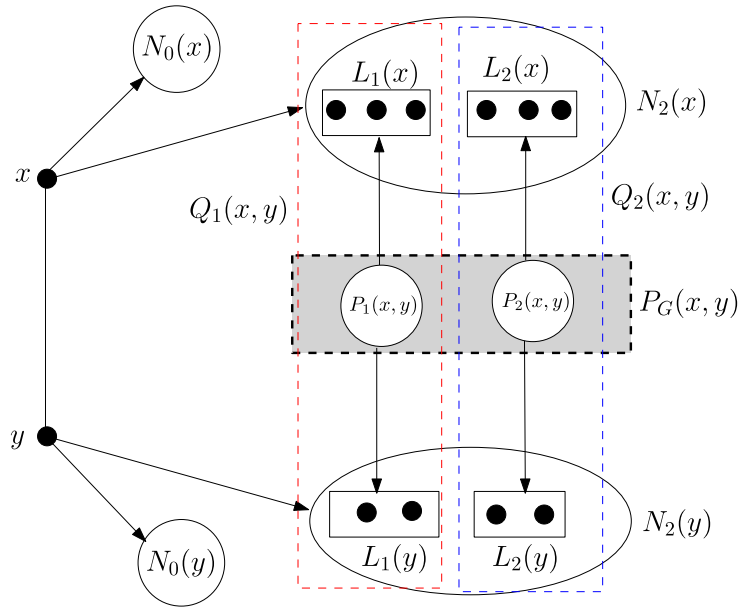


Fig. 3. Ricci curvature of graphs with girth at least 5: structure of the core neighborhood.

Case 1:

$f(y) = -1$. This implies that $f(z) \leq 0$ for $z \in N_G(y)$ and $f(z) \geq -1$, for $z \in N_G(x)$. Therefore, from Eq. (20) we get

$$E_y(f) - E_x(f) \leq \frac{1}{d_x} + \frac{|N_0(x)| + |N_2(x)|}{d_x} = 1.$$

Moreover, this bound is attained by the function $g(z) := -1 \cdot \mathbf{1}_{\{z \in N_G(x) \cup \{y\}\}}$ which is 1-Lipschitz on the core neighborhood of (x, y) . This implies, $\kappa_{-1}(x, y) = 0$.

Case 2:

$f(y) = 0$. This implies that $f(z) \leq 1$ for $z \in N_G(y)$ and $f(z) \geq -1$, for $z \in N_G(x)$. Therefore, from Eq. (20) we get,

$$\begin{aligned} E_y(f) - E_x(f) &\leq \frac{|N_0(y)| + |N_2(y)|}{d_y} + \frac{|N_0(x)| + |N_2(x)|}{d_x}, \\ &= 2 - \frac{1}{d_x} - \frac{1}{d_y}. \end{aligned} \tag{21}$$

This implies that $\kappa_0(x, y) \geq -1 + \frac{1}{d_x} + \frac{1}{d_y}$.

Now, consider the 1-Lipschitz function $g : V(G_{(x,y)}) \mapsto \mathbb{R}$

$$g(z) = \begin{cases} -1, & \text{if } z \in N_0(x) \cup N_2(x); \\ 1, & \text{if } z \in N_0(y) \cup N_2(y); \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $E_y(g) - E_x(g) = 2 - \frac{1}{d_x} - \frac{1}{d_y}$. Therefore, g attains the lower bound in Eq. (22) and we have

$$\kappa_0(x, y) = -1 + \frac{1}{d_x} + \frac{1}{d_y}. \tag{22}$$

Case 3:

$f(y) = 1$. This implies that $f(z) \leq 2$ for $z \in N_G(y)$ and $f(z) \geq -1$, for $z \in N_G(x)$. From Eq. (20) and similar to Case 3 in Theorem 3.1 we get,

$$E_y(f) - E_x(f) \leq \frac{2|N_0(y)|}{d_y} + \frac{|N_0(x)| - 1}{d_x} + \sum_{a=1}^q \left\{ \frac{W_f(L_a(y))}{d_y} - \frac{W_f(L_a(x))}{d_x} \right\}, \tag{23}$$

where $L_a(x)$ and $L_a(y)$ for $a \in \{1, 2, \dots, q\}$ are as defined in statement of the theorem. As before, for any 1-Lipschitz function $f : V(G_{(x,y)}) \mapsto \mathbb{Z}$ with $f(x) = 0, f(y) = 1$ denote the restriction of f to $L_a(y) \cup L_a(x) \cup \{x, y\}$ as f_a , for $a \in \{1, 2, \dots, q\}$.

Define $\mathcal{W}(f_a)$ as before. Then it is easy to see that $g_a := \arg \max_{f_a \in \mathcal{L}_1, f_a(y)=1} \mathcal{W}(f_a)$ must be either

- (i) $g_a(z) = 1$ for all $z \in L_a(y)$ and $g_a(z) = -1$ for all $z \in L_a(x)$,
- (ii) $g_a(z) = 2$ for all $z \in L_a(y)$ and $g_a(z) = 0$ for all $z \in L_a(x)$.

This implies that,

$$\begin{aligned} \max_{\substack{f_a \in \mathcal{L}_1, \\ f_a(y)=0}} \mathcal{W}(f_a) &= \max \left\{ \frac{|L_a(y)|}{d_y} + \frac{|L_a(x)|}{d_x}, \frac{2|L_a(y)|}{d_y} \right\}, \\ &= \left\{ \frac{|L_a(y)|}{d_y} + \frac{|L_a(x)|}{d_x} \right\} \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} \geq \frac{|L_a(y)|}{d_y} \right\}} + \frac{2|L_a(y)|}{d_y} \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} < \frac{|L_a(y)|}{d_y} \right\}}, \\ &= \left\{ \frac{|L_a(x)|}{d_x} - \frac{|L_a(y)|}{d_y} \right\} \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} \geq \frac{|L_a(y)|}{d_y} \right\}} + \frac{2|L_a(y)|}{d_y}. \end{aligned} \quad (24)$$

Noting that $\sum_{a=1}^q |L_a(y)| = |N_2(y)|$, and substituting Eq. (24) in Eq. (23) we get

$$\begin{aligned} \kappa_1(x, y) &\geq 1 - \frac{|N_0(x)| - 1}{d_x} - \frac{2(|N_0(y)| + |N_2(y)|)}{d_y} - \sum_{a=1}^q \left\{ \frac{|L_a(x)|}{d_x} - \frac{|L_a(y)|}{d_y} \right\} \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} \geq \frac{|L_a(y)|}{d_y} \right\}}, \\ &= -2 + \frac{2}{d_x} + \frac{2}{d_y} + \frac{|N_2(x)|}{d_x} - \sum_{a=1}^q \left\{ \frac{|L_a(x)|}{d_x} - \frac{|L_a(y)|}{d_y} \right\} \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} \geq \frac{|L_a(y)|}{d_y} \right\}}. \end{aligned} \quad (25)$$

Now, consider the 1-Lipschitz function $g : V(G_{(x,y)}) \mapsto \mathbb{R}$ with $g(x) = 0$, $g(y) = 1$ and

$$g(z) = \begin{cases} -1, & \text{if } z \in N_0(x); \\ 2, & \text{if } z \in N_0(y); \\ g_a(z), & \text{if } z \in L_a(x) \cup L_a(y). \end{cases},$$

where for $a \in \{1, 2, \dots, q\}$,

$$g_a(z) := \begin{cases} -1 \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} \geq \frac{|L_a(y)|}{d_y} \right\}}, & \text{if } z \in L_a(x); \\ 1 \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} \geq \frac{|L_a(y)|}{d_y} \right\}} + 2 \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} < \frac{|L_a(y)|}{d_y} \right\}}, & \text{if } z \in L_a(y); \\ 1 \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} < \frac{|L_a(y)|}{d_y} \right\}}, & \text{if } z \in P_G(x, y). \end{cases}$$

It is easy to see that g attains the lower bound in Eq. (25) and we have

$$\kappa_1(x, y) = -2 + \frac{2}{d_x} + \frac{2}{d_y} + \frac{|N_2(x)|}{d_x} - \sum_{a=1}^q \left\{ \frac{|L_a(x)|}{d_x} - \frac{|L_a(y)|}{d_y} \right\} \cdot \mathbf{1}_{\left\{ \frac{|L_a(x)|}{d_x} \geq \frac{|L_a(y)|}{d_y} \right\}}. \quad (26)$$

Note that $\kappa(x, y) = \kappa_{-1}(x, y) \wedge \kappa_0(x, y) \wedge \kappa_1(x, y) = \kappa_0(x, y) \wedge \kappa_1(x, y)$, since $\kappa_{-1}(x, y) = 0$, and the proof completes. As in Remark 3.1, it is easy to check that (19) is symmetric in x and y . \square

3.3. Extension to weighted graphs

The above theorems can be readily extended to weighted graphs.

3.3.1. Weighted bipartite graphs

Since Lemma 2.3 also holds for weighted graphs, we can replace G by its core neighborhood $G_{(x,y)}$, and the structure of the core neighborhood $G_{(x,y)}$ is as depicted in Fig. 2. For any $Z \subseteq V(G_{(x,y)})$ and any function $f \in \mathcal{L}_1$, denote by $W_f^x(Z) = \sum_{z \in Z} f(z)w_{xz}$ and $W_f^y(Z) = \sum_{z \in Z} f(z)w_{yz}$. Let $N_0(x)$, $N_0(y)$, $N_1(x)$, and $N_1(y)$ be as defined before Theorem 3.1. Therefore, for $(x, y) \in E(G)$,

$$E_y(f) - E_x(f) = \frac{f(x)w_{xy} + W_f^y(N_0(y)) + W_f^y(N_1(y))}{d_y} - \frac{f(y)w_{xy} + W_f^x(N_0(x)) + W_f^x(N_1(x))}{d_x}, \quad (27)$$

where $d_x = \sum_{(x,y) \in E(G)} w_{xy}$, for $x \in V(G)$. Lemma 2.2 implies that it suffices to maximize $E_y(f) - E_x(f)$ over 1-Lipschitz functions f satisfying $f(x) = 0$, and $f(y) \in \{-1, 0, 1\}$. Now, by calculations identical to those in the proof of Theorem 3.1 it can be shown that:

Corollary 3.4. Let $G = (V(G), E(G))$ be a locally finite weighted bipartite graph and $(x, y) \in E(G)$. Suppose $R(x, y)$ is the subgraph of $G_{(x,y)}$ induced by $N_1(x) \cup N_1(y)$, and $R_1(x, y), R_2(x, y), \dots, R_q(x, y)$ be the connected components of $R(x, y)$. If $U_a(x) = V(R_a(x, y)) \cap N_1(x)$ and $U_a(y) = V(R_a(x, y)) \cap N_1(y)$, for $a \in \{1, 2, \dots, q\}$, then

$$\kappa(x, y) = -2 \left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \frac{\sum_{z \in N_1(y)} w_{yz}}{d_y} + \sum_{a=1}^q \left(\frac{\sum_{z \in U_a(y)} w_{yz}}{d_y} - \frac{\sum_{z \in U_a(x)} w_{xz}}{d_x} \right) \right)_+ \tag{28}$$

3.3.2. Weighted graphs with girth greater than 4

Similarly, by calculations identical to those in the proof of Theorem 3.1, it is possible to obtain the Ricci-curvature for locally finite weighted graphs with girth at least 5.

Corollary 3.5. Let $G = (V(G), E(G))$ be a locally finite weighted graph with girth $g(G) \geq 5$ and $(x, y) \in E(G)$. Suppose $Q(x, y)$ is the subgraph of $G_{(x,y)}$ induced by $N_2(x) \cup N_2(y) \cup P_G(x, y)$, and $Q_1(x, y), Q_2(x, y), \dots, Q_q(x, y)$ be the connected components of $Q(x, y)$. If $L_a(x) = V(Q_a(x, y)) \cap N_2(x)$ and $L_a(y) = V(Q_a(x, y)) \cap N_2(y)$, for $a \in \{1, 2, \dots, q\}$, then $\kappa(x, y) = \kappa_0(x, y) \wedge \kappa_1(x, y)$, where

$$\kappa_0(x, y) = - \left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} \right)_+ \tag{29}$$

and

$$\kappa_1(x, y) = -2 \left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \frac{\sum_{z \in N_2(x)} w_{xz}}{2d_x} + \frac{1}{2} \sum_{a=1}^q \left(\frac{\sum_{z \in L_a(x)} w_{xz}}{d_x} - \frac{\sum_{z \in L_a(y)} w_{yz}}{d_y} \right) \right)_+ \tag{30}$$

4. Ricci-flat graphs of girth 5

The notion of Ricci-flat graphs was first introduced by Chung and Yau [5]. It was defined for regular graphs and was used to prove logarithmic Harnack inequalities for graphs. These inequalities are useful in bounding the Log-Sobolev constant, and, consequently, the mixing times of Markov chains [7]. Recently, Lin et al. [17,16] defined Ricci-flat graphs as graphs where the modified Ricci-curvature vanishes on every edge. This definition is motivated by the fact that Ricci-flat manifolds are Riemannian manifolds where the Ricci curvature vanishes. Moreover, this definition does not require the graph to be regular. Ricci flat graphs defined by Chung and Yau are not necessarily Ricci-flat in this sense (for constructions of Ricci-flat graphs under both definitions refer to Lin et al. [17]).

In this paper, we consider Ricci-flat graphs with respect to Olliver’s coarse Ricci curvature, that is, a graph G is said to be Ricci-flat if $\kappa(x, y) = 0$, for all $(x, y) \in E(G)$. The following corollary characterizes Ricci-flat graphs with girth at least 5. Hereafter, only locally finite simple unweighted graphs are considered.

Corollary 4.1. A connected graph G is a Ricci-flat graph with $g(G) \geq 5$, if and only if G is one of the following: the path P_n ($n \geq 2$), the infinite ray, the infinite path, the cycle C_n ($n \geq 5$), or the star graph T_n ($n \geq 3$).

Proof. Suppose $\kappa(x, y) = 0$ for all $(x, y) \in E(G)$ and $g(G) \geq 5$. Then it follows from Theorem 3.3 that $\frac{1}{d_x} + \frac{1}{d_y} \geq 1$, for all $(x, y) \in E(G)$. This implies that either $d_x = d_y = 2$, or $d_x \wedge d_y = 1$, for all $(x, y) \in E(G)$. The following two cases arise:

- Case 1: There is a vertex $v \in V(G)$ with $d_v \geq 3$. Then all neighbors of v must have degree 1, with no edges left to connect to other vertices. Thus, the graph G must be a n -star T_n rooted at v .
- Case 2: $d_x \leq 2$ for all $x \in V(G)$. If $d_x = 2$, for all $x \in V(G)$, then it is easy to see that G is an infinite path or a cycle of length at least 5. Hence, it suffices to assume that there is at least one vertex $V(G)$ with degree 1. Then there can be at most 2 vertices with degree 1. Thus, depending on whether the number of degree 1 vertices is one or two, G is either the infinite half ray or the finite path P_n , respectively. \square

Lin et al. [17] characterized Ricci-flat graphs of girth 5 using their modified definition of Ricci-curvature. The above corollary is the analogous version of their result using Ollivier’s original definition of coarse Ricci-curvature. As it happens, the structure of Ollivier’s Ricci-flat graphs of girth 5 is much simpler than the structure of the modified Ricci-flat graphs. Apart from the infinite path and the cycle, the modified Ricci-flat graphs of girth 5 include quite complicated graphs, such as the Peterson graph, the dodecahedral graph, and the half-dodecahedral graph. Ollivier’s Ricci-flat graph however includes the n -star, which is not included in the modified definition. This illustrates that the structure of Ricci-flat graphs in the two definitions of Ricci-curvature are, in fact, quite different.

5. A general lower bound with maximum matching and Ricci-flat regular graphs of girth 4

In this section we establish connections between Ricci-curvature and the size of the matching in the core neighborhood subgraph. We prove a general lower bound on the Ricci-curvature in terms of the size of the maximum matching. The bound is often nearly tight, especially in triangle-free regular graphs. Using this bound we prove a necessary and sufficient condition on the structure of Ricci-flat regular graphs of girth 4.

5.1. A general lower bound with maximum matching

Let $G = (V, E)$ be a fixed graph, with $(x, y) \in E(G)$. Recall that $N_G(x)$ and $N_G(y)$ denote the set of neighbors of x and y , and $\Delta_G(x, y) = N_G(x) \cap N_G(y)$. Define, $Q_G(x) = N_G(x) \setminus \Delta_G(x, y)$ and $Q_G(y) = N_G(y) \setminus \Delta_G(x, y)$. Let $\overline{H}_G(x, y)$ be the subgraph of G induced by the vertices in $Q_G(x) \cup Q_G(y)$. A matching in $\overline{H}_G(x, y)$ is a collection of disjoint edges $(a, b) \in E(G)$, with $a \in Q_G(x)$ and $b \in Q_G(y)$.

Theorem 5.1. *Let $G = (V, E)$ be a fixed graph, with $(x, y) \in E(G)$. If $|M_G(x, y)| (\leq |Q_G(x) \wedge |Q_G(y)|)$ is the size of the maximum matching in $\overline{H}_G(x, y)$, then*

$$\kappa(x, y) \geq \frac{|\Delta_G(x, y)|}{d_x \vee d_y} - 2 \left(1 - \frac{|M_G(x, y)| + |\Delta_G(x, y)|}{d_x \vee d_y} \right).$$

Moreover, if $|M_G(x, y)| = |Q_G(x) \wedge |Q_G(y)|$, then $\kappa(x, y) \geq \frac{|\Delta_G(x, y)|}{d_x \vee d_y} - 2 \left(1 - \frac{d_x \wedge d_y}{d_x \vee d_y} \right)$.

Proof. W.l.o.g. assume $d_x \leq d_y$ and let $|M_G(x, y)| = k$. By the definition of matching there exists $T_x := \{a_1, a_2, \dots, a_k\} \subseteq Q_G(x)$ and $T_y := \{b_1, b_2, \dots, b_k\} \subseteq Q_G(y)$ such that $d_G(a_i, b_i) = 1$. Consider any 1-Lipschitz function $f : V(G) \mapsto \mathbb{Z}$. Lemma 2.2 implies that it suffices to optimize over $f(x) = 0$. This means that $|f(z)| \leq 1$ for all $z \in N_G(x)$, and $|f(a_i) - f(b_i)| \leq 1$.

$$\begin{aligned} T_{xy}(f) &= \sum_{i=1}^k \left(\frac{f(b_i)}{d_y} - \frac{f(a_i)}{d_x} \right) + \sum_{z \in \Delta_G(x, y)} \left(\frac{f(z)}{d_y} - \frac{f(z)}{d_x} \right) + \sum_{z \in Q_G(y) \setminus T_y} \frac{f(z)}{d_y} - \sum_{z \in Q_G(x) \setminus T_x} \frac{f(z)}{d_x} \\ &\leq \frac{k}{d_y} + \left(\frac{1}{d_x} - \frac{1}{d_y} \right) \left\{ \sum_{i=1}^k |f(a_i)| + \sum_{z \in \Delta_G(x, y)} |f(z)| \right\} + \sum_{z \in Q_G(y) \setminus T_y} \frac{f(z)}{d_y} + \frac{|Q_G(x) \setminus T_x|}{d_x} \\ &\leq \frac{k}{d_y} + \left(\frac{1}{d_x} - \frac{1}{d_y} \right) \{k + |\Delta_G(x, y)|\} + \frac{2|Q_G(y) \setminus T_y|}{d_y} + \frac{|Q_G(x) \setminus T_x|}{d_x} \\ &= \frac{k}{d_y} + \left(\frac{1}{d_x} - \frac{1}{d_y} \right) \{k + |\Delta_G(x, y)|\} + \frac{2(|Q_G(y)| - k)}{d_y} + \frac{|Q_G(x)| - k}{d_x}. \end{aligned} \tag{31}$$

The result follows from noting that $\kappa(x, y) = 1 - \sup_{f \in \mathcal{L}_1, f(x)=0} T_{xy}(f)$, and $|Q_G(x)| = d_x - |\Delta_G(x, y)|$, $|Q_G(y)| = d_y - |\Delta_G(x, y)|$. If $d_x \leq d_y$ and $k = |Q_G(x) \wedge |Q_G(y)|$, then $k = |Q_G(x)| = d_x - |\Delta_G(x, y)|$, and the result follows from the previous bound by direct substitution. \square

Combining the lower bound in the above theorem and the upper bound from Theorem 1.1 we get

$$\frac{|\Delta_G(x, y)|}{d_x \vee d_y} - 2 \left(1 - \frac{|M_G(x, y)| + |\Delta_G(x, y)|}{d_x \vee d_y} \right) \leq \kappa(x, y) \leq \frac{|\Delta_G(x, y)|}{d_x \vee d_y}. \tag{32}$$

In the following section we obtain a necessary and sufficient condition on the structure of regular graphs for which the upper bound in Eq. (32) is tight. As a consequence we characterize Ricci-flat regular graphs with girth 4.

5.2. Ricci-flat regular graphs with girth 4

Recall that the Birkhoff polytope B_n is the convex polytope in \mathbb{R}^{n^2} whose points are the doubly stochastic matrices, that is, the $n \times n$ matrices whose entries are non-negative real numbers and whose rows and columns each add up to 1 [31]. The Birkhoff–von Neumann theorem states that the extreme points of the Birkhoff polytope are the permutation matrices, that is, matrices with exactly one entry 1 in each row and each column and 0 elsewhere.

Using this result on the Birkhoff polytope and Theorem 5.1 it is easy to get a necessary and sufficient condition on the structure of Ricci-flat graphs with girth 4 which are regular. In fact, we shall prove a much stronger result where we give a necessary and sufficient condition on the structure of graphs for which the upper bound in Eq. (32) is an equality.

Theorem 5.2. *For be a graph $G = (V, E)$, with $(x, y) \in E(G)$ and $d_x = d_y = d$, $\kappa(x, y) = \frac{|\Delta_G(x, y)|}{d}$, if and only if there is a perfect matching between $Q_G(x)$ and $Q_G(y)$.*

Proof. If there is a perfect matching between $Q_G(x)$ and $Q_G(y)$, then by Theorem 5.1, $\kappa(x, y) = \frac{\Delta_G(x,y)}{d}$.

Conversely, suppose $\kappa(x, y) = \frac{\Delta_G(x,y)}{d}$. From Eq. (3)

$$\kappa(x, y) = 1 - \inf_{\nu \in \mathcal{A}} \sum_{z_1 \in N_G(x)} \sum_{z_2 \in N_G(y)} \nu(z_1, z_2) d(z_1, z_2),$$

where \mathcal{A} is the set of all $d \times d$ matrices with entries indexed by $N_G(x) \times N_G(y)$ such that $\nu(x', y') \geq 0$, $\sum_{z \in N_G(y)} \nu(x', z) = \frac{1}{d}$, and $\sum_{z \in N_G(x)} \nu(z, y') = \frac{1}{d}$, for all $x' \in N_G(x)$ and $y' \in N_G(y)$. Therefore, \mathcal{A} forms a Birkhoff polytope in \mathbb{R}^{d^2} (after multiplying with d), and since $\kappa(x, y)$ is a linear function defined over \mathcal{A} , it is maximized at one of the extreme points. Therefore, by the Birkhoff–von Neumann theorem the optimal transfer plan is a permutation matrix.

To complete the proof, note that the optimal transfer plan, which is given by a permutation matrix, cannot transfer any mass to or from $\Delta_G(x, y)$. Moreover, each transfer must be over a path of length 1. This is because a mass of $1/d$ needs to be transferred by a path of length at least 1 for all vertices in $N_G(x) \setminus \Delta_G(x, y)$, which already gives $W_1(x, y) \geq 1 - \Delta_G(x, y)/d$, and so any further mass transfer will result in $\kappa(x, y) < \Delta_G(x, y)/d$, a contradiction. This implies that there must be a perfect matching between $Q_G(x)$ and $Q_G(y)$. \square

When G is triangle free, by definition $Q_G(x) = N_G(x)$ and $Q_G(y) = N_G(y)$. The following corollary is then immediate from the above theorem:

Corollary 5.3. *A connected graph G with $g(G) = 4$ and $d_x = d_y = d$ has $\kappa(x, y) = 0$ if and only if there is a perfect matching between $N_G(x)$ and $N_G(y)$.*

This implies that regular, triangle-free Ricci-flat graphs must have a perfect matching between $N_G(x)$ and $N_G(y)$ for all $(x, y) \in E(G)$. The n -dimensional integer lattice \mathbb{Z}^n , the n -dimensional hypercube C_n^n , a cycle C_n of length $n \geq 4$, and the complete bipartite graph $K_{n,n}$, are examples of regular Ricci-flat graphs of girth 4. Identifying the set of all such regular graphs remains open.

6. Ricci curvature of random graphs

In this section we study the behavior of Ollivier’s Ricci-curvature for Erdős–Rényi random graphs $\mathbb{G}(n, p)$ in different regimes of p . As seen in the previous section, the Ricci-curvature depends on the size of matchings in the core neighborhood subgraph. In this section we will prove a technical matching lemma, establish properties of matchings in random bipartite graphs, and use these results to obtain Ricci-curvature of random graphs.

6.1. A more technical matching lemma

Let $G = (V, E)$ be a fixed graph, with $(x, y) \in E(G)$. Define, $R_G(x) = (N_G(x) \setminus \{y\}) \setminus \Delta_G(x, y)$ and $R_G(y) = (N_G(y) \setminus \{x\}) \setminus \Delta_G(x, y)$. Let $H_G(x, y)$ be the subgraph of G induced by the vertices in $R_G(x) \cup R_G(y)$. The subgraph $H_G(x, y)$ is said to have a m -matching of size k if there exists $T_x := \{a_1, a_2, \dots, a_k\} \subseteq R_G(x)$ and $T_y := \{b_1, b_2, \dots, b_k\} \subseteq R_G(y)$ such that $d_G(a_i, b_i) \leq m$, for $m \in \mathbb{Z}^+$ and $k \leq |R_G(x)| \wedge |R_G(y)|$. Note that a 1-matching of size k is just the standard bipartite matching of size k between $R_G(x)$ and $R_G(y)$ in the subgraph $H_G(x, y)$.

The following lemma gives a lower bound on the Ricci-curvature in terms of the size of 2-matchings in $H_G(x, y)$.

Lemma 6.1. *Let $G = (V, E)$ be a fixed graph, with $(x, y) \in E(G)$, and $H_G(x, y)$ be the subgraph of G induced by the vertices in $R_G(x) \cup R_G(y)$. If there exists a 2-matching of size $k (\leq |R_G(x)| \wedge |R_G(y)|)$ in the $H_G(x, y)$, then*

$$\kappa(x, y) \geq -2 + \frac{3|\Delta_G(x, y)| + k + 2}{d_x \vee d_y}.$$

Moreover, if $k = |R_G(x)| \wedge |R_G(y)|$, then $\kappa(x, y) \geq -2 + \frac{2|\Delta_G(x, y)| + d_x \wedge d_y + 1}{d_x \vee d_y}$.

Proof. W.l.o.g. assume $d_x \leq d_y$. By the definition of m -matching there exists $T_x := \{a_1, a_2, \dots, a_k\} \subseteq R_G(x)$ and $T_y := \{b_1, b_2, \dots, b_k\} \subseteq R_G(y)$ such that $d_G(a_i, b_i) \leq 2$. Consider any 1-Lipschitz function $f : V(G) \mapsto \mathbb{Z}$. Lemma 2.2 implies that it suffices to optimize over $f(x) = 0$. This means that $|f(z)| \leq 1$ for all $z \in N_G(x)$, and $|f(a_i) - f(b_i)| \leq 2$. If $T_{xy}(f) = \mathbb{E}_x(f) - \mathbb{E}_y(f)$, then from calculations similar to the proof of Theorem 5.1

$$\begin{aligned} T_{xy}(f) &= -\frac{f(y)}{d_x} + \sum_{i=1}^k \left(\frac{f(b_i)}{d_y} - \frac{f(a_i)}{d_x} \right) + \sum_{z \in \Delta_G(x, y)} \left(\frac{f(z)}{d_y} - \frac{f(z)}{d_x} \right) + \sum_{z \in R_G(y) \setminus T_y} \frac{f(z)}{d_y} - \sum_{z \in R_G(x) \setminus T_x} \frac{f(z)}{d_x} \\ &\leq \frac{|f(y)|}{d_x} + \frac{2k}{d_y} + \left(\frac{1}{d_x} - \frac{1}{d_y} \right) \{k + |\Delta_G(x, y)|\} + \frac{2(|R_G(y)| - k)}{d_y} + \frac{|R_G(x)| - k}{d_x}. \end{aligned} \tag{33}$$

Note that $\kappa(x, y) = 1 - \sup_{f \in \mathcal{L}_1, f(x)=0} T_{xy}(f)$. Therefore, simplifying Eq. (33) the result follows. Moreover, if $d_x \leq d_y$ and $k = |R_G(x)| \wedge |R_G(y)|$, then $k = |R_G(x)| = d_x - 1 - |\Delta_G(x, y)|$, and the result follows from the previous bound by direct substitution. \square

6.2. Matchings in random bipartite graphs

Matchings in random graphs are well studied in the literature [12], beginning with the celebrated result of Erdős and Rényi [10] on the existence of perfect matchings. The proof of this result relies on the Hall’s marriage theorem. Here we use a variant of Hall’s theorem to obtain an analogous result about the existence of near-perfect matching in random bipartite graphs.

Recall, Hall’s marriage theorem states that a bipartite graph $G = (V, E)$ with bipartition (A, B) has a perfect matching if and only if for all $X \subseteq A$, $|N_G(X)| \geq |X|$, where $N_G(X) = \bigcup_{x \in X} N_G(x)$. We will need the following strengthening of Hall’s theorem [9]:

Theorem 6.1 ([9]). Consider a bipartite graph $G = (V, E)$ with bipartition (A, B) . For $X \subseteq A$, define $\delta(X) = |X| - |N_G(X)|$, where $N_G(X) = \bigcup_{x \in X} N_G(x)$. Let $\delta_{\max} = \max_{X \subseteq A} \delta(X)$. Then the size of the maximum matching in G is $|A| - \delta_{\max}$. \square

Using this theorem we now prove the following lemma about the existence of near-perfect matching in random bipartite graphs.

Lemma 6.2. For every $\varepsilon \in (0, 1)$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\mathbb{G}(n, n, p) \text{ has a matching of size } n(1 - \varepsilon)) = \begin{cases} 0 & \text{if } np \rightarrow 0, \\ 1 & \text{if } np \rightarrow \infty. \end{cases}$$

Proof. Let $G \sim \mathbb{G}(n, n, p)$ be a random bipartite graph with bipartition (A, B) , with $|A| = |B| = n$ and edge probability p . If $np \rightarrow 0$ and there is a matching of size $n(1 - \varepsilon)$ in G , then $|E(G)| \geq n(1 - \varepsilon)$. But $E(G) \sim \text{Bin}(n^2, p)$, and so $\mathbb{P}(E(G) \geq n(1 - \varepsilon)) \rightarrow 0$ by Markov’s inequality.

Next, suppose $np \rightarrow \infty$. Let $X \subseteq A$, with $|X| \geq \varepsilon n$. If $D(X) = \sum_{x \in X} d_x$, then $D(X) \sim \text{Bin}(n|X|, p)$, and by Hoeffding’s inequality $\mathbb{P}(D(X) < |X|) \leq \exp(-2|X|(np - 1)^2)$. Thus, by a union bound,

$$\begin{aligned} \mathbb{P}(\exists X \subseteq A : |N_G(X)| < |X|, |X| \geq \varepsilon n) &\leq \mathbb{P}(\exists X \subseteq A : D(X) < |X|, |X| \geq \varepsilon n) \\ &\leq \sum_{|X|=\varepsilon n}^n \binom{n}{|X|} \exp(-2|X|(np - 1)^2) \\ &\leq 2^n \exp(-2\varepsilon n(np - 1)^2) \rightarrow 0. \end{aligned}$$

Thus, with probability $1 - o(1)$, for any $X \subseteq A$, with $|X| \geq \varepsilon n$, $|N_G(X)| \geq |X|$. Therefore, from Theorem 6.1, G has a matching of size $n(1 - \varepsilon)$ with probability $1 - o(1)$, when $np \rightarrow \infty$. \square

Remark 6.1. Lemma 6.2 immediately gives that for a random bipartite graph $\mathbb{G}(m, n, p)$ with $(m \wedge n)p \rightarrow \infty$, $\lim_{m, n \rightarrow \infty} \mathbb{P}(\mathbb{G}(m, n, p) \text{ has a matching of size } (m \wedge n)(1 - \varepsilon)) = 1$.

6.3. Ricci curvature of random bipartite graphs

We are now ready to state and prove our result on the Ricci curvature of random bipartite graphs. Let $\mathbb{G}(n, n, p)$ be a random bipartite graph with bipartition (A_n, B_n) . Let $a \in A_n$ and $b \in B_n$ be two fixed vertices. For $G \sim \mathbb{G}(n, n, p)$, conditioned on the edge (a, b) being present, denote by $\kappa_n(a, b)$ the Ricci-curvature of the edge (a, b) in G .

Theorem 6.2. Let $G \sim \mathbb{G}(n, n, p_n)$ be a random bipartite graph with bipartition (A_n, B_n) , conditioned on the edge (a, b) being present.

- (a) If $np_n \rightarrow 0$ then with probability $1 - o(1)$ the edge (a, b) is isolated, and consequently $\kappa_n(a, b) = 0$.
- (b) If $np_n \rightarrow \lambda$ for $0 < \lambda < \infty$, then $\kappa_n(a, b) \xrightarrow{\mathcal{D}} -2 \left(1 - \frac{1}{1+X_1} - \frac{1}{1+X_2}\right)_+$, where X_1, X_2 are independent $\text{Poisson}(\lambda)$ random variables, that is, the Ricci curvature converges in distribution to Ricci-curvature of its limiting tree.
- (c) If $np_n \rightarrow \infty$, and $np_n^2 \rightarrow 0$ then $\kappa_n(a, b) \xrightarrow{\mathcal{D}} -2$.
- (d) If $np_n^2 \rightarrow \infty$, then $\kappa_n(a, b) \xrightarrow{\mathcal{D}} 0$.

Proof. Let $G \sim \mathbb{G}(n, n, p_n)$ be a random bipartite graph with bipartition $A_n = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $B_n = \{\beta_1, \beta_2, \dots, \beta_n\}$ conditioned on the edge (a, b) being present. For $i, j \in \{1, 2, \dots, n\} := [n]$ define let $\delta_{ij} = 1$ if $(\alpha_i, \beta_j) \in E(G)$ and 0, otherwise. Define $X_n^a := \sum_{j \in [n] \setminus \{b\}} \delta_{aj}$ and $X_n^b := \sum_{i \in [n] \setminus \{a\}} \delta_{ib}$. Clearly, (X_n^a, X_n^b) are independent binomial random variables with parameters $(n - 1, p_n)$, which are measurable with respect to $\mathcal{F}_n := \sigma(\delta_{aj}, \delta_{ib}, i \in [n] \setminus \{a\}, j \in [n] \setminus \{b\})$.

- (a) In this case, $\mathbb{P}(\{X_n^a \neq 0\} \cup \{X_n^b \neq 0\}) \leq \mathbb{E}(X_n^a + X_n^b) \leq 2np_n \rightarrow 0$. Therefore, with probability $1 - o(1)$ the edge (a, b) is isolated.
- (b) Observe that, in this case,

$$\mathbb{P}\left(\sum_{i \in [n] \setminus \{b\}} \sum_{j \in [n] \setminus \{a\}} \delta_{aj} \delta_{ji} \delta_{ib} \neq 0\right) \leq \sum_{i \in [n] \setminus \{b\}} \sum_{j \in [n] \setminus \{a\}} \mathbb{E}(\delta_{aj} \delta_{ji} \delta_{ib}) \leq n^2 p_n^3 = O(1/n).$$

Therefore, with probability $1 - O(1/n)$ there are no 4-cycles in G supported on (a, b) . As G is bipartite, the girth of G is at least 6, and Corollary 2.2 implies that

$$\kappa_n(a, b) = -2 \left(1 - \frac{1}{1 + X_n^a} - \frac{1}{1 + X_n^b}\right)_+.$$

The conclusion follows on noting that $(X_n^a, X_n^b) \xrightarrow{\mathcal{D}} (X_1, X_2)$, where X_1, X_2 are independent $Poisson(\lambda)$ random variables.

- (c) Let $R_G(b)$ denote the set of vertices in $N_G(b)$ which are not connected to any vertex in $N_G(a)$. Then given \mathcal{F}_n ,

$$|R_G(b)| \sim Bin(X_n^b, 1 - (1 - p_n)^{X_n^a}) \quad \text{and} \quad \mathbb{E}\left(\frac{|R_G(b)|}{X_n^b} \middle| \mathcal{F}\right) = 1 - (1 - p_n)^{X_n^a} \xrightarrow{\mathcal{D}} 0,$$

as $np_n^2 \rightarrow 0$. Consequently, we have $|R_G(b)| = o_p(X_n^b)$. Defining $R_G(a)$ by symmetry we have $|R_G(a)| = o_p(X_n^a)$. Now, consider the 1-Lipschitz function f on the (random) core-neighborhood of G defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in N_G(a) \setminus R_G(a), \\ 1 & \text{if } x \in R_G(a) \cup \{a\}, \\ 2 & \text{if } x \in R_G(b) \cup \{b\}, \\ 3 & \text{if } x \in N_G(b) \setminus R_G(b). \end{cases}$$

This implies that

$$\mathbb{E}_b(f) - \mathbb{E}_a(f) = \frac{1 + 2|R_G(b)| + 3(X_n^b - |R_G(b)|)}{1 + X_n^b} - \frac{2 + |R_G(a)|}{1 + X_n^a}.$$

The RHS converges to 3 in probability. Thus, $W_n(a, b) \geq 3 - o(1)$ with probability $1 - o(1)$. Moreover, by definition $W_n(a, b) \leq 3$ and the conclusion follows.

- (d) As bipartite graphs are triangle free, from Eq. (6) we know that $\kappa(a, b) \leq 0$. So, it suffices to prove only the lower bound.

To this effect, note that on the set $C_n := \{X_n^a \geq np_n/2, X_n^b \geq np_n/2\}$ we have $(X_n^a \wedge X_n^b)p_n \geq np_n^2/2 \rightarrow \infty$. Thus, if we denote by $H_G(a, b)$ the random bipartite graph induced by $N_G(a) \cup N_G(b)$, by Remark 6.1 we have

$$\mathbb{P}(H_G(a, b) \text{ does not have a matching of size } (1 - \varepsilon)(|N_G(a)| \wedge |N_G(b)|) \mid \mathcal{F}_n, C_n) \leq \delta,$$

for n large enough. Let E_n be the event $\{H_G(a, b) \text{ does not have a matching of size } (1 - \varepsilon)(|N_G(a)| \wedge |N_G(b)|)\}$. This implies that for n large enough

$$\mathbb{P}(E_n) \leq \mathbb{E}(\mathbb{P}(E_n \mid \mathcal{F}_n, C_n)) + \mathbb{P}(C_n^c) \leq \delta + \mathbb{P}(\{X_n^a \wedge X_n^b \geq np_n/2\}^c).$$

As $(X_n^a \wedge X_n^b)/np_n \xrightarrow{\mathcal{D}} 1$, when $np_n \rightarrow \infty$, the above implies that with probability $1 - o(1)$ there exists a bipartite matching between in $H_G(a, b)$ of size $(1 - \varepsilon)(|N_G(a)| \wedge |N_G(b)|)$. This observation, together with Theorem 5.1, would imply that with probability $1 - o(1)$ we have

$$\lim_{n \rightarrow \infty} \kappa_n(a, b) \geq -2 \left(1 - \frac{(X_n^a \wedge X_n^b)(1 - \varepsilon) + 1}{(1 + X_n^a) \vee (1 + X_n^b)}\right) = -2\varepsilon.$$

As ε is arbitrary, we have $\kappa_n(a, b) \xrightarrow{\mathcal{D}} 0$. \square

6.4. Ricci curvature of Erdős–Rényi random graphs

Building on the techniques developed in the previous section, we now determine the limiting behavior of Ricci-curvature for Erdős–Rényi random graphs $\mathbb{G}(n, p_n)$ in different regimes of p_n .

Theorem 6.3. *Let $G \sim \mathbb{G}(n, p_n)$ be a Erdős–Rényi random graph with vertex set V_n , conditioned on the edge (a, b) being present.*

- (a) If $np_n \rightarrow 0$ then with probability $1 - o(1)$ the edge (a, b) is isolated, and consequently $\kappa_n(a, b) = 0$.
- (b) If $np_n \rightarrow \lambda$ for $0 < \lambda < \infty$, then $\kappa_n(a, b) \xrightarrow{\mathcal{D}} -2 \left(1 - \frac{1}{1+X_1} - \frac{1}{1+X_2}\right)_+$, where X_1, X_2 are independent $Poisson(\lambda)$ random variables, that is, the Ricci curvature converges in distribution to Ricci-curvature of its limiting tree.
- (c) If $np_n \rightarrow \infty, n^2 p_n^3 \rightarrow 0$, then $\kappa_n(a, b) \xrightarrow{\mathcal{D}} -2$.

- (d) If $n^2 p_n^3 \rightarrow \infty, np_n^2 \rightarrow 0$, then $\kappa_n(a, b) \xrightarrow{\mathcal{P}} -1$.
- (e) If $np_n^2 \rightarrow \infty, p_n \rightarrow 0$, then $\kappa_n(a, b) \xrightarrow{\mathcal{P}} 0$.
- (f) If $p_n \rightarrow p$ with $0 < p < 1$, then $\kappa_n(a, b) \xrightarrow{\mathcal{P}} p$.

Proof. Let $V_n = \{v_1, v_2, \dots, v_n\}$ and $G \sim \mathbb{G}(n, p_n)$ be a random graph with vertex set V_n , conditioned on the edge (a, b) being present, for 2 fixed vertices $a, b \in V_n$. For $\{i, j\} \subseteq \{1, 2, \dots, n\} := [n]$ define let $\delta_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and 0, otherwise. Define $X_n^a := \sum_{j \in [n] \setminus \{a, b\}} \delta_{aj}$ and $X_n^b := \sum_{i \in [n] \setminus \{a, b\}} \delta_{ib}$. Clearly, (X_n^a, X_n^b) are independent binomial random variables with parameters $(n - 2, p_n)$, which are measurable with respect to $\mathcal{F}_n := \sigma(\delta_{aj}, \delta_{ib}, i \in [n] \setminus \{a, b\}, j \in [n] \setminus \{a, b\})$.

- (a) By Markov’s inequality we have $\mathbb{P}(\{X_n^a + X_n^b \neq 0\}) \leq \mathbb{E}(X_n^a + X_n^b) \leq 2np_n \rightarrow 0$. So, the edge (a, b) is isolated with probability $1 - o(1)$, and $\kappa_n(a, b) = 0$.
- (b) We will show that in this case there are no cycles of length 3, 4, or 5 supported on (a, b) with probability $1 - O(1/n)$. In this regard, note that

$$\begin{aligned} \mathbb{P}(\exists \text{ a 3-cycle supported on } (a, b)) &\leq \sum_{j \in [n] \setminus \{a, b\}} \mathbb{E}(\delta_{aj} \delta_{bj}) \leq np_n^2 = O(1/n), \\ \mathbb{P}(\exists \text{ a 4-cycle supported on } (a, b)) &\leq \sum_{\{j, k\} \subset [n] \setminus \{a, b\}} \mathbb{E}(\delta_{aj} \delta_{jk} \delta_{bk}) \leq n^2 p_n^3 = O(1/n), \\ \mathbb{P}(\exists \text{ a 5-cycle supported on } (a, b)) &\leq \sum_{\{j, k, l\} \subset [n] \setminus \{a, b\}} \mathbb{E}(\delta_{aj} \delta_{jk} \delta_{kl} \delta_{bl}) \leq n^3 p_n^4 = O(1/n). \end{aligned}$$

This implies that $\mathbb{P}(\exists \text{ a 3, 4, or 5 cycle supported on } (a, b)) = O(1/n)$. Therefore, with probability $1 - O(1/n)$, the girth of G is at least 6, and Corollary 2.2 implies that

$$\kappa_n(a, b) = -2 \left(1 - \frac{1}{1 + X_n^a} - \frac{1}{1 + X_n^b} \right)_+.$$

The conclusion follows on noting that $(X_n^a, X_n^b) \xrightarrow{\mathcal{P}} (X_1, X_2)$, where X_1, X_2 are independent $Poisson(\lambda)$ random variables.

- (c) As in the previous case, the probability of a triangle and a quadrilateral supported on (a, b) is bounded by np_n^2 and $n^2 p_n^3$, and so with probability $1 - o(1)$ there are no 3 and 4-cycles supported on (a, b) . Let $Q_2(b) = \{x \in N_G(b) : d_G(x, N_G(a)) = 2\}$. Note that

$$|Q_2(b)| \Big| \mathcal{F}_n \sim Bin(X_n^b, 1 - (1 - p_n^2)^{(n-4)X_n^a}) \quad \text{and} \quad \mathbb{E} \left(\frac{|Q_2(b)|}{X_n^b} \Big| \mathcal{F}_n \right) = 1 - (1 - p_n^2)^{(n-4)X_n^a} \xrightarrow{\mathcal{P}} 0.$$

Now, consider the 1-Lipschitz function f on the (random) core-neighborhood of G defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in N_G(a), \\ 2 & \text{if } x \in Q_2(b) \cup \{b\}, \\ 3 & \text{if } x \in N_G(b) \setminus Q_2(b), \\ 1 & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbb{E}_b(f) - \mathbb{E}_a(f) = \frac{1 + 2|Q_2(b)| + 3(X_n^b - |Q_2(b)|)}{1 + X_n^b} - \frac{2}{1 + X_n^a}.$$

The RHS converges to 3 in probability, and the conclusion follows from arguments similar to the proof of part (c) of Theorem 6.2.

- (d) In this case there are no triangles supported on (a, b) with probability $1 - o(1)$. Define $R_1(a) = \{x \in N_G(a) : d_G(x, N_G(b)) = 1\}$, and $R_1(b)$ similarly. By an argument similar to the previous case we have $R_1(a) = o_p(X_n^a)$ and $R_1(b) = o_p(X_n^b)$. Now, for any $z_1 \in N_G(a) \setminus R_1(a)$ and $z_2 \in N_G(b) \setminus R_1(b)$ we have $d(z_1, z_2) \geq 2$, and so the function f defined below is 1-Lipschitz:

$$f(x) = \begin{cases} 0 & \text{if } x \in N_G(a) \cup \{a\}, \\ 2 & \text{if } x \in N_G(b) \setminus R_1(b), \\ 1 & \text{otherwise.} \end{cases}$$

Plugging in f we have

$$\mathbb{E}_a(f) - \mathbb{E}_b(f) = \frac{|R_2(b)| + 2(X_n^b - |R_2(b)|)}{1 + X_n^b} - \frac{1}{1 + X_n^a}.$$

The RHS converges to 2 in probability, proving the upper bound for $\kappa_n(a, b)$.

To prove the lower bound, assuming $\Delta_G(a, b) = \emptyset$ we claim that with probability $1 - o(1)$ there exists a 2-matching between $N_G(a)$ and $N_G(b)$ of size $(1 - \varepsilon)(X_n^a \wedge X_n^b)$. Assuming the claim is true by Lemma 6.1

$$\kappa_n(a, b) \geq -2 + \frac{2 + (1 - \varepsilon)X_n^a \wedge X_n^b}{(1 + X_n^a) \vee (1 + X_n^b)}.$$

The RHS converges to $-1 - \varepsilon$ in probability, which would finish the proof of (d).

To complete the proof we need to verify the claim, that is, construct a 2-matching between $N_G(a)$ and $N_G(b)$ in the subgraph $H_G(a, b)$. Define a new random graph $\mathcal{H}(G)$ with $V(\mathcal{H}(G)) = N_G(a) \cup N_G(b)$ and an edge between $i \in N_G(a)$ and $j \in N_G(b)$ if and only if in the original random graph G there is a path from i to j of length 2.

By this construction, the probability that there is an edge between i and j in $\mathcal{H}(G)$ is $\hat{p}_n := 1 - (1 - p_n^2)^{n-4}$. By a Taylor’s expansion we have

$$|1 - (1 - p_n^2)^{n-4} - np_n^2| \leq 4p_n^2 + \frac{1}{2}n^2p_n^4(1 + p_n^2)^{n-6} = np_n^2 \left(\frac{4}{n} + \frac{1}{2}np_n^2e^{np_n^2} \right) = np_n^2o(1),$$

and so for all large n we have $\hat{p}_n \geq np_n^2/2$. Note that there is a 2-matching in $H_G(a, b)$ of size $(1 - \varepsilon)(X_n^a \wedge X_n^b)$ if and only if $\mathcal{H}(G)$ has a 1-matching of size $(1 - \varepsilon)(X_n^a \wedge X_n^b)$. Since existence of a matching is a monotone property and the edges in $\mathcal{H}(G)$ are positively correlated, w.l.o.g. we may assume that the edges in $\mathcal{H}(G)$ are independent, as that would further reduce the probability of a matching. Now, on the set $C_n := \{X_n^a \geq np_n/2, X_n^b \geq np_n/2\}$ we have $(X_n^a \wedge X_n^b)\hat{p}_n \geq np_n\hat{p}_n/2 \geq n^2p_n^3/4 \rightarrow \infty$. Thus, by Remark 6.1 we have

$$\mathbb{P}(\mathcal{H}(G) \text{ has no matching of size } (1 - \varepsilon)(X_n^a \wedge X_n^b) | \mathcal{F}_n, C_n, |\Delta_G(a, b)| = 0) \leq \delta,$$

from which the claim follows, as $\mathbb{P}(C_n) \rightarrow 1$.

- (e) In this case there may be triangles supported on (a, b) . Recall $R_G(a) = N_G(a) \setminus \Delta_G(a, b)$ and $R_G(b) = N_G(b) \setminus \Delta_G(a, b)$, and $H_G(a, b)$ is the subgraph of G induced by the vertices in $R_G(a) \cup R_G(b)$. This implies that $|R_G(a)| = X_n^a - |\Delta_G(a, b)|$, $|R_G(b)| = X_n^b - |\Delta_G(a, b)|$. As $[(X_n^a \wedge X_n^b) - |\Delta_G(a, b)|]p_n \xrightarrow{\mathcal{P}} \infty$, by Remark 6.1 there exists a matching in $H_G(a, b)$ of size $(1 - \varepsilon)(X_n^a \wedge X_n^b - \Delta_G(a, b))$ with probability $1 - o(1)$, and so by Theorem 5.1 we have

$$\kappa_n(a, b) \geq \frac{|\Delta_G(a, b)|}{(1 + X_n^a) \vee (1 + X_n^b)} - 2 \left(1 - \frac{(X_n^a \wedge X_n^b - \Delta_G(a, b))(1 - \varepsilon) + |\Delta_G(a, b)|}{(1 + X_n^a) \vee (1 + X_n^b)} \right).$$

The RHS converges to -2ε in probability, as $|\Delta_G(x, y)| = o_p(X_n^a \wedge X_n^b)$. This proves that $\kappa_n(a, b) \geq 0$ in probability. Moreover, by Eq. (6) we have $\kappa_n(a, b) \leq \frac{|\Delta_G(x, y)|}{(X_n^a + 1) \wedge (X_n^b + 1)}$ which converges to 0 in probability, completes the proof.

- (f) By a similar argument as in the previous case, there exists a matching between $R_G(a)$ and $R_G(b)$ of size $(1 - \varepsilon)(X_n^a \wedge X_n^b - |\Delta_G(a, b)|)$, and so by Theorem 5.1 we have

$$\kappa_n(a, b) \geq \frac{|\Delta_G(a, b)|}{(1 + X_n^a)(1 + X_n^b)} - 2 \left(1 - \frac{(X_n^a \wedge X_n^b - |\Delta_G(a, b)|)(1 - \varepsilon) + |\Delta_G(a, b)|}{(1 + X_n^a) \vee (1 + X_n^b)} \right).$$

The RHS converges to $p - 2\varepsilon(1 - p)$, which together with Eq. (6) implies that $\kappa_n(a, b) \xrightarrow{\mathcal{P}} p$. \square

7. Conclusions

In this paper we derive exact formulas for Ollivier’s Ricci-curvature for bipartite graphs and graphs with girth at least 5. We also prove a general lower bound on $\kappa(x, y)$ in terms of the size of the matching among the non-common neighbors of x and y in the graph G . This bound is often tight, especially in regular graphs which have a perfect matching between the non-common neighbors. This together with the Birkhoff–von Neumann provide a necessary and sufficient condition on the structure of Ricci-flat regular graphs of girth 4. However, it remains open to characterize the set of graphs with a perfect matching between the non-common neighbors of x and y , for all edges (x, y) in the graph.

We also study the Ricci-curvature of random graphs and characterize the limiting behavior of the Ricci-curvature in the regimes where it has a constant limit in probability. Studying Ricci-curvature for other random graph models is an interesting problem for future research.

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References

- [1] D. Bakry, Functional inequalities for Markov semigroups, in: S.G. Dani, P. Graczyk (Eds.), *Probability Measures on Groups: Recent Directions and Trends*, Tata, Bombay, 2006, pp. 91–147.
- [2] D. Bakry, M. Émery, Diffusions hypercontractives, in: *Séminaire de probabilités, XIX, 1983/84*, in: *Lecture Notes in Math.*, vol. 1123, Springer, Berlin, 1985, pp. 177–206.
- [3] F. Bauer, J. Jost, S. Liu, Ollivier–Ricci curvature and the spectrum of the normalized graph Laplace operator, *Math. Res. Lett.* 19(6) (2012) 1185–1205.
- [4] H.J. Cho, S.-H. Paeng, Ollivier’s Ricci curvature and the coloring of graphs, *European J. Combin.* 34 (2013) 916–922.
- [5] Fan Chung, S.-T. Yau, Logarithmic Harnack inequalities, *Math. Res. Lett.* 3 (1996) 793–812.
- [6] P. Diaconis, The Markov chain Monte Carlo revolution, *Bull. Amer. Math. Soc. (N.S.)* 46 (2009) 179–205.
- [7] P. Diaconis, L. Saloff-Coste, Logarithmic Sobolev inequalities for finite Markov chains, *Ann. Appl. Probab.* 6 (3) (1996) 695–750.
- [8] P. Diaconis, D. Stroock, Geometric bounds for eigenvalues of Markov chains, *Ann. Appl. Probab.* 1 (1991) 36–61.
- [9] R. Diestel, *Graph Theory*, fourth ed., in: *Graduate Texts in Mathematics*, vol. 173, Springer-Verlag, Heidelberg, 2010.
- [10] P. Erdős, A. Rényi, On the existence of a factor of degree one of a connected random graph, *Acta Math. Acad. Sci. Hungar.* 17 (3–4) (1966) 359–368.
- [11] J. Fulman, E. Wilmer, Comparing eigenvalue bounds for Markov chains: when does Poincare beat Cheeger? *Ann. Appl. Probab.* 9 (1999) 1–13.
- [12] S. Janson, T. Luczak, A. Ruciński, *Random Graphs*, Wiley, New York, 2000.
- [13] J. Jost, S. Liu, Ollivier’s Ricci curvature, local clustering and curvature dimension inequalities on graphs, *Discrete Comput. Geom.* 51 (2014) 300–322.
- [14] A. Joulin, Y. Ollivier, Curvature, concentration and error estimates for Markov chain Monte Carlo, *Ann. Probab.* 38 (6) (2010) 2418–2442.
- [15] D.A. Levin, Y. Peres, E.L. Wilmer, *Markov Chains and Mixing Times*, Amer. Math. Soc., Providence, RI, 2009.
- [16] Y. Lin, L. Lu, S.T. Yau, Ricci curvature of graphs, *Tohoku Math. J.* 63 (4) (2011) 605–627.
- [17] Y. Lin, L. Lu, S.T. Yau, Ricci-flat graphs with girth at least five, *Comm. Anal. Geom.* 22 (4) (2014) 1–17.
- [18] Y. Lin, S.T. Yau, Ricci curvature and eigenvalue estimate on locally finite graphs, *Math. Res. Lett.* 17 (2) (2010) 343–356.
- [19] B. Loisel, P. Romon, Ricci curvature on polyhedral surfaces via optimal transportation, *Axioms* 3 (1) (2014) 119–139.
- [20] J. Lott, C. Villani, Ricci curvature for metric measure spaces via optimal transport, *Ann. of Math.* 169 (3) (2009) 903–991.
- [21] S. Ohta, On measure contraction property of metric measure spaces, *Comment. Math. Helv.* 82 (2007) 805–828.
- [22] Y. Ollivier, Ricci curvature of metric spaces, *C. R. Math. Acad. Sci. Paris* 345 (11) (2007) 643–646.
- [23] Y. Ollivier, Ricci curvature of Markov chains on metric spaces, *J. Funct. Anal.* 256 (3) (2009) 810–864.
- [24] Y. Ollivier, A survey of Ricci curvature for metric spaces and Markov chains, in: *Probabilistic Approach to Geometry*, in: *Adv. Stud. Pure Math.*, vol. 57, Math. Soc. Japan, Tokyo, 2010, pp. 343–381.
- [25] S.-H. Paeng, Volume and diameter of a graph and Ollivier’s Ricci curvature, *European J. Combin.* 33 (2012) 1808–1819.
- [26] A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, in: *Algorithms and Combinatorics*, vol. 24, Springer, Berlin, 2003.
- [27] K.-T. Sturm, On the geometry of metric measure spaces, I, *Acta Math.* 196 (2006) 65–131.
- [28] K.-T. Sturm, On the geometry of metric measure spaces, II, *Acta Math.* 196 (2006) 133–177.
- [29] C. Villani, *Topics in Optimal Transportation*, in: *Graduate Studies in Mathematics*, vol. 58, American Mathematical Society, Providence, RI, 2003.
- [30] Translated by V.A. Yemelichev, M.M. Kovlev, M.K. Kravtsov, in: G.H. Lawden (Ed.), *Polytopes, Graphs, Optimization*, Cambridge University Press, Cambridge, 1984.
- [31] G.M. Ziegler, *Lectures on Polytopes*, first ed., in: *Graduate Texts in Mathematics*, vol. 152, Springer, New York, 2007.