BSDE Approach to Non-Zero-Sum Stochastic Differential Games of Control and Stopping *

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Dedicated to Professor Robert J. Elliott on the Occasion of his 70th Birthday

Abstract

This paper studies two non-zero-sum stochastic differential games of control and stopping. One game has interaction in the players' stopping rules, whereas the other does not. Solutions to backward stochastic differential equations (BSDEs) will be shown to provide the value processes of the first game. A multi-dimensional BSDE with reflecting barrier is studied in two cases for its solution: existence and uniqueness with Lipschitz growth, and existence in a Markovian system with linear growth rate. The extension to linear/quadratic growth rates of the equation allows the controls to observe the instantaneous volatilities of the value processes in the Markovian case.

Keywords and Phrases: Stochastic differential game, Nash equilibrium, optimal control, optimal stopping, backward stochastic differential equation.

AMS 2000 Subject Classifications: Primary 93E20, 60G40; secondary 91A06/23/55.

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1 Introduction

1.1 Bibliographic notes

We study non-zero-sum stochastic differential games, in which each of several players chooses an optimal strategy to maximize his own reward - usually the sum of a running reward and a terminal reward. The rewards of the players do not have to add up to a constant (say, zero); that is, the game is not necessarily zero-sum. The objective is to find an equilibrium point; namely, if every player's strategy is his best response to the other players' strategies, then the set of strategies is called an equilibrium point of the game, conventionally a Nash equilibrium. It is an equilibrium in the sense that no player will profit by unilaterally changing his strategy, when all the other players' strategies remain the same. The terminology "Nash equilibrium" is in deference to J. Nash's formulation of this notion of equilibrium in 1949. This notion of equilibrium for a non-zero-sum game generalizes the Von Neumann-Morgenstern notion of saddle point for a zero-sum game.

In the literature on both zero-sum and non-zero-sum stochastic differential games of control, the existence and the choice of optimal controls are shown to be equivalent to the existence of controls that satisfy the *Isaacs condition,* which is the attainability of the maxima or maxima of the associated Hamiltonians. In contrast to the counterpart optimization problem studied by Beneš in [1], [2] and by Davis (1973) in [9], the Isaacs (or Nash) condition for N-player non-zero-sum games does not typically hold. The existence, or non-existence, of an optimal control set that maximizes the Hamiltonians serves as an easy-to-verify local condition equivalent to the existence, or non-existence, of equilibrium controls.

With Markovian rewards, which are functions of the current value of an underling diffusion state process, partial differential equations are a handy tool. Over the past thirty years, Bensoussan, Frehse and Friedman built a regularity theory of PDE's to study stochastic differential games. Among their extensive works, Bensoussan and Friedman (1977) considered in [5] games of optimal stopping. The existence of optimal stopping times for such games is reduced to the study of regular solutions to quasi-variational inequalities, assuming continuous and bounded running rewards and terminal rewards. Bensoussan and Frehse (2000) in [3] solved a non-zero-sum game of optimal controls, which is terminated when the state process exits a bounded domain. Their running rewards are quadratic forms of the controls. The monograph by Fleming and Soner (1993) [17] gives a full account of controlled Markov diffusions.

As a tool for stochastic control theory, backward stochastic differential equations (BSDEs for short) were first proposed by Bismut in the 1970's. The connection between the two subjects can be viewed as a stochastic version of the verification theorems for the control of Markov diffusions. Pardoux and Peng (1990) proved in [34] existence and uniqueness of the solution to a BSDE with uniformly Lipschitz growth. Considerable attention has been devoted to studying the association between BSDEs and stochastic differential games. Cvitanić and Karatzas (1996) proved in [8] existence and uniqueness of the solution to the equation with double reflecting barriers, and associated their BSDE to a zero-sum Dynkin game. Their work generalized El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) [12] on one-dimensional BSDE with one reflecting boundary, which captures early stopping features as that of American options. Hamadène and Lepeltier (2000) [22] and Hamadène (2006) [23] added controls to the Dynkin game studied by Cvitanić and Karatzas (1996) [8], the tool still being BSDE with double reflecting barriers. Markovian rewards of games correspond to the equations in the Markovian framework. Hamadène studied in [20] and [21] Nash equilibrium control with forward-backward SDE. In Hamadène, Lepeltier and Peng (1997) [19], the growth rates of their forward-backward SDE are linear in the value process and the volatility process, and polynomial in the state process. Their state process is a diffusion satisfying an "$\mathbb{L}^2$-dominance" condition. These three authors solve a non-zero-sum game without stopping, based on existence result of the multi-dimensional BSDE.

The martingale method also facilitates the study of zero-sum and non-zero-sum games, and is particularly useful if the rewards depend on the path of the state process. This method is surveyed by Davis (1979) [10]. Elliott (1976) [14] shows that the Isaacs condition implies the existence of value and saddle strategies for a zero-sum game of control. When there are terminal rewards only, Lepeltier and Etourneau (1987) in [32] used martingale techniques to provide sufficient conditions for the existence of optimal stopping times on processes that need not be Markovian; their general theory requires some order assumption and supermartingale assumptions on the terminal reward. Karatzas and Zamfirescu (2008) in [31] took the martingale approach to characterize, then construct, saddle points for zero-sum games of control and stopping. They also characterized the value processes by the semimartingale decompositions and proved a stochastic maximum principle for continuous, bounded running reward that can be a functional of the diffusion state process.

Zero-sum games of stopping (games of timing, Dynkin games) are connected to singular controls, in the sense that, for convex cost functions, the value functions of the former games are derivatives of the value functions of the latter. This connection was first observed by Taksar (1985) [35], followed by Fukushima and Taksar (2002) [18] in a Markovian setting via solving free-boundary problems, and by Karatzas and Wang (2001) [30] in a non-Markovian setting based on weak compactness arguments.

1.2 This paper

This paper considers a non-zero-sum game with features of both stochastic control and optimal stopping, for a process of diffusion type via the backward SDE approach. Running rewards, terminal rewards and early exercise rewards are all included. The running rewards can be functionals of the diffusion state process. Since the Nash equilibrium of an N-player non-zero-sum game is technically not more difficult than a two-player non-zero-sum game, only notationally more tedious, the number of players is assumed to be two, for concreteness.

Section 2 solves for the existence of Nash equilibrium for the stochastic games of control and stopping. The controls enter the drift of the underlying state process. Each player controls and stops the reward streams. In Game 2.1, a player's choice of stopping time terminates his own reward stream only. The value processes of both players are part of the solution to a multi-dimension BSDE with reflecting barrier. The instantaneous volatilities of the two players' value processes are explicitly expressed in the solution. Existence of the solution to general forms of the multi-dimensional BSDE with reflecting barrier will be proven in section 3 and section 4. Then, in the Markovian framework, the instantaneous volatilities can enter the controls as arguments, in which case the game is said to *observe volatilities* in addition to the other two arguments, namely time and the state-process. In Game 2.2, each player can terminate the game not only for himself but also for the other player(s). We shall establish the existence of an equilibrium in a weaker sense than that for Game 2.1. It seems beyond the reach of our current ability, to develop a more general theory with the present methods.

Section 3 proves existence and uniqueness of the solution to a multi-dimensional BSDE with reflecting barrier, a general form of the one that accompanies Game 2.1. Section 4 discusses extension of the existence of solutions to equations of ultra-Lipschitz growth.

The BSDE approach here proposes a multi-dimensional BSDE whose value processes in the solution provide the value processes of the non-zero-sum games. General existence result of solutions to multi-dimensional BSDE with reflecting barrier still need to be developed. As is proven in Hu and Peng (2006) [25], in several dimensions, the comparison theorem holds only under very restrictive conditions. Cohen, Elliott and Pearce (2010) [16] recently give a general component-wise condition for comparison of multi-dimensional BSDEs. Without invoking comparison results or penalization methods, we use Picard iterations to show the existence of solutions to equations with Lipschitz drivers. When the growth condition is ultra-Lipschitz, convergence arguments of the usual Picard type iteration cannot proceed, either. In a Markovian framework, this paper proves the Markovian structure of solutions to multi-dimensional reflected BSDEs with Lipschitz growth, and uses this Markovian structure as a starting point to extend existence results to equations with growth rates which are linear in the value and volatility processes, and polynomial in the state process. Once again, the method does not rely on comparison theorems. The linear/quadratic growth rates of the equations allow the controls to observe the instantaneous volatilities of the players' value processes.

Our multi-dimensional BSDEs with reflection differ from the multi-dimensional BSDEs with oblique reflection in Hu and Tang (2010) [26], in Hamadène and Zhang (2010) [24] and in Chassagneux, Elie and Kharroubi (2010) [6], which are associated with optimal switching problems. Our equations are generalizations of the doubly reflected BSDEs in Cvitanić and Karatzas (1996) [8], which are associated with Dynkin games. The difference of these equations are determined by the essential difference of control problems, zero-sum games and non-zero-sum games.

We have tried to separate the game aspect and the BSDE aspect in the write-up of the paper, so we organize it in such a way that readers interested in stochastic differential games can read section 2 only, ignoring BSDE technicalities; whereas BSDE connoisseurs can examine section 3 and section 4.

With BSDEs as one of its tools, the theory of Stochastic Control has extensive applications in the rich fields of Mathematical Finance and Economics, like pricing and hedging of contingent claims, portfolio optimization, risk management, algorithmic trading, utility maximization, and so on. Many of these subjects are discussed in the survey paper El Karoui, Peng and Quenez (1997) [13]. The literature on Mathematical Finance tends to focus on the optimal behavior of one agent, or at most on zero-sum games between a buyer and a seller. Real-world financial phenomena, on the other hand, result inherently from the interaction of several agents, who try to optimize their own profits and whose actions form a non-zero-sum game. The present work can be seen as a contribution to the mathematical foundations of the study of the interaction among several such agents in financial markets.

2 The games of Control and Stopping

In the non-zero sum games of control and stopping to be discussed in this paper, each player receives a reward. Based on their up-to-date information, the two players I and II, respectively, first choose their controls u and v, then the times τ and ρ to stop their own reward streams. The controls u and v are two processes that enter the dynamics of the underlying state process for the rewards. The optimality criterion for our non-zero-sum games is that of a Nash equilibrium, in which each player's expected reward is maximized when the other player maximizes his. In taking conditional expectations of the rewards, the change-of-measure setup to be formulated fixes one single Brownian filtration and one single state process for all controls u and v. Hence when optimizing the expected rewards over the control sets, there is no need to keep in mind the filtration or the state process.

Let us set up the rigorous model. We start with a d-dimensional Brownian motion $B(\cdot)$ with respect to its generated filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$ on the canonical probability space $(\Omega, \mathscr{F}, \mathbb{P})$, in which $\Omega = C^d[0,T]$ is the set of all continuous d-dimensional function on a finite deterministic time horizon $[0,T]$, $\mathscr{F} = \mathscr{B}\left(C^d[0,T]\right)$ is the Borel sigma algebra, and $\mathbb{P}$ is the Wiener measure.

For every $t \in [0,T]$, define a mapping $\phi_t : C[0,T] \to [0,T]$ by $\phi_t(y)(s) = y(s\wedge t)$, which truncates the function $y \in C[0,T]$. For any $y^0 \in C[0,T]$, the pre-image $\phi_t^{-1}(y^0)$ collects all functions in $C[0,T]$ which are identical to y^0 up to time t. A stopping rule is a mapping $\tau : C[0,T] \to [0,T]$, such that

$$\{y \in C[0,T] : \tau(y) \leq t\} \in \phi_t^{-1}\left(\mathscr{B}\left(C[0,T]\right)\right). \tag{2.1}$$

The set of all stopping rules ranging between t_1 and t_2 is denoted by $\mathscr{S}(t_1, t_2)$.

In the **path-dependent** case, the state process $X(\cdot)$ solves the stochastic functional equation

$$X(t) = X(0) + \int_0^t \sigma(s, X)dB_s,\ 0 \leq t \leq T, \tag{2.2}$$

where the volatility matrix $\sigma : [0,T]\times\Omega \to \mathbb{R}^d \times \mathbb{R}^d$, $(t,\omega) \mapsto \sigma(t,\omega)$, is a predictable process. In particular in the **Markovian** case, the volatility matrix $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is a deterministic mapping $[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto \sigma(t,x) \in \mathbb{R}^d \times \mathbb{R}^d$, and then the state process equation (2.2) becomes the stochastic differential equation

$$X(t) = X(0) + \int_0^t \sigma(s, X(s)) dB_s, \; 0 \le t \le T. \tag{2.3}$$

The Markovian case is indeed a special case of path-dependence. Since it will receive some extra attention later at the end of subsection 2.2, we describe the Markovian framework separately from the more general path-dependent case.

Assumption 2.1 *(1) The volatility matrix $\sigma(t,\omega)$ is nonsingular for every $(t,\omega) \in [0,T] \times \Omega$;*
(2) there exists a positive constant A such that

$$|\sigma_{ij}(t,\omega) - \sigma_{ij}(t,\bar{\omega})| \le A \sup_{0 \le s \le t} |\omega(s) - \bar{\omega}(s)|, \tag{2.4}$$

for all $1 \le i,j \le d$, for all $t \in [0,T]$, $\omega, \bar{\omega} \in \Omega$.

Under Assumption 2.1 (2), for every initial value $X(0) \in \mathbb{R}^d$, there exists a pathwise unique strong solution to equation (2.2) (Theorem 14.6, Elliott (1982) [15]).

The controls u and v take values in some given separable metric spaces $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively. We shall assume that $\mathbb{A}_1$ and $\mathbb{A}_2$ are countable unions of nonempty, compact subsets of these spaces, and are endowed with the σ-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ of their respective Borel subsets. The controls u and v are said
(i) to be **open loop**, if $u_t = \mu(t,\omega)$ and $v_t = \nu(t,\omega)$ are $\{\mathscr{F}_t\}_{0 \le t \le T}$-adapted processes on $[0,T]$, where $\mu : [0,T] \times \Omega \to \mathbb{A}_1$ and $\nu : [0,T] \times \Omega \to \mathbb{A}_2$ are non-anticipative measurable mappings;
(ii) to be **closed loop**, if $u_t = \mu(t,X)$ and $v_t = \nu(t,X)$ are non-anticipative functionals of the state process $X(\cdot)$, for $0 \le t \le T$, where $\mu : [0,T] \times \Omega \to \mathbb{A}_1$ and $\nu : [0,T] \times \Omega \to \mathbb{A}_2$ are deterministic measurable mappings;
(iii) to be **Markovian**, if $u_t = \mu(t,X(t))$ and $v_t = \nu(t,X(t))$, for $0 \le t \le T$, where $\mu : [0,T] \times \mathbb{R}^d \to \mathbb{A}_1$ and $\nu : [0,T] \times \mathbb{R}^d \to \mathbb{A}_2$ are deterministic measurable functions.

In the path-dependent case, the set $\mathscr{U} \times \mathscr{V}$ of admissible controls are taken as all the closed loop controls. The techniques that we shall use to solve for the optimal closed loop controls also apply to the open loop controls, so the extension of the results from closed loop to open loop is only a matter of more complicated notation. The discussion will be restricted within the class of closed loop controls for clarity of the exposition. In the Markovian case, the set $\mathscr{U} \times \mathscr{V}$ of admissible controls are taken as all the Markovian controls. The collection of Markovian controls is a subset of the collection of closed loop controls.

We consider the predictable mapping

$$\begin{aligned} f : [0,T] \times \Omega \times \mathbb{A}_1 \times \mathbb{A}_2 &\to \mathbb{R}^d, \\ (t,\omega,\mu(t,\omega),\nu(t,\omega)) &\mapsto f(t,\omega,\mu(t,\omega),\nu(t,\omega)), \end{aligned} \tag{2.5}$$

in the path-dependent case, and the deterministic measurable mapping

$$
\begin{aligned}
&f:[0,T]\times\Omega\times\mathbb{A}_1\times\mathbb{A}_2\to\mathbb{R}^d,\\
&(t,\omega,\mu(t,\omega(t)),\nu(t,\omega(t)))\mapsto f(t,\omega(t),\mu(t,\omega(t)),\nu(t,\omega(t))),
\end{aligned}
\tag{2.6}
$$

in the Markovian case, satisfying:

Assumption 2.1 *(continued)*
(3) There exists a positive constant A such that

$$
\left|\sigma^{-1}(t,\omega)f(t,\omega,\mu(t,\omega),\nu(t,\omega))\right|\leq A, \tag{2.7}
$$

and

$$
|\sigma(t,\omega)|^2\leq A\sup_{0\leq s\leq t}\left(1+|\omega(s)|^2\right), \tag{2.8}
$$

for all $0\leq t\leq T$, $\omega\in\Omega$, and for all the $\mathbb{A}_1\times\mathbb{A}_2$-valued representative elements $(\mu(t,\omega),\nu(t,\omega))$ of the control spaces $\mathscr{U}\times\mathscr{V}$.

For generic controls $u_t=\mu(t,\omega)$ and $v_t=\nu(t,\omega)$, we define $\mathbb{P}^{u,v}$, a probability measure equivalent to $\mathbb{P}$, via the Radon-Nikodym derivative

$$
\left.\frac{d\mathbb{P}^{u,v}}{d\mathbb{P}}\right|_{\mathscr{F}_t}=\exp\left\{\int_0^t\sigma^{-1}(s,X)f(s,X,u_s,v_s)dB_s-\frac{1}{2}\int_0^t|\sigma^{-1}(s,X)f(s,X,u_s,v_s)|^2ds\right\} \tag{2.9}
$$

for $0\leq t\leq T$. Then, by the Girsanov Theorem,

$$
B_t^{u,v}:=B_t-\int_0^t\sigma^{-1}(s,X)f(s,X,u_s,v_s)ds,\ 0\leq t\leq T \tag{2.10}
$$

is a $\mathbb{P}^{u,v}$-Brownian Motion on $[0,T]$ with respect to the filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$. In the Markovian case, the expression of (2.10) can be written as

$$
B_t^{u,v}=B_t-\int_0^t\sigma^{-1}(s,X(s))f(s,X(s),\mu(s,X(s)),\nu(s,X(s)))ds,\ 0\leq t\leq T. \tag{2.11}
$$

On the probability space $(\Omega,\mathscr{F},\mathbb{P})$, and with respect to the filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$, the pair $(X,B^{u,v})$ is a weak solution to the forward stochastic functional equation

$$
X(t)=X(0)+\int_0^t f(s,X,u_s,v_s)ds+\int_0^t\sigma(s,X)dB_s^{u,v},\ 0\leq t\leq T \tag{2.12}
$$

in the path-dependent case, and a weak solution to the forward stochastic differential equation

$$
X(t)=X(0)+\int_0^t f(s,X(s),\mu(s,X(s)),\nu(s,X(s)))ds+\int_0^t\sigma(s,X(s))dB_s^{u,v},\ 0\leq t\leq T \tag{2.13}
$$

in the Markovian case.

When playing the game, the two players choose first their admissible controls u in $\mathscr{U}$ and v in $\mathscr{V}$, then for any given $t \in [0,T]$, they chose τ_t and ρ_t from $\mathscr{S}(t,T)$, times for them to quit the game. The pair of control and stopping rule (u,τ) is up to player I and the pair (v,ρ) is up to player II. For starting the game at time t, applying controls u and v, and quitting the game at τ_t and ρ_t respectively, the players receive rewards $R^1_t(\tau_t,\rho_t,u,v)$ and $R^2_t(\tau_t,\rho_t,u,v)$. To take into account the uncertainty inherent in the situation they face, their respective reward processes are measured by the conditional $\mathbb{P}^{u,v}$-expectations

$$\mathbb{E}^{u,v}[R^1_t(\tau_t,\rho_t,u,v)|\mathscr{F}_t] \quad \text{and} \quad \mathbb{E}^{u,v}[R^2_t(\tau_t,\rho_t,u,v)|\mathscr{F}_t]. \tag{2.14}$$

In the non-zero-sum games, the two players seek first admissible control strategies u^* in $\mathscr{U}$ and v^* in $\mathscr{V}$, and then stopping rules τ^*_t and ρ^*_t from $\mathscr{S}(t,T)$, to maximize their expected rewards, in the sense that

$$\begin{aligned}
&\mathbb{E}^{u^*,v^*}[R^1_t(\tau^*_t,\rho^*_t,u^*,v^*)|\mathscr{F}_t] \geq \mathbb{E}^{u,v^*}[R^1_t(\tau_t,\rho^*_t,u,v^*)|\mathscr{F}_t],\ \forall \tau_t \in \mathscr{S}(t,T),\ \forall u \in \mathscr{U};\\
&\mathbb{E}^{u^*,v^*}[R^2_t(\tau^*_t,\rho^*_t,u^*,v^*)|\mathscr{F}_t] \geq \mathbb{E}^{u^*,v}[R^2_t(\tau^*_t,\rho_t,u^*,v)|\mathscr{F}_t],\ \forall \rho_t \in \mathscr{S}(t,T),\ \forall v \in \mathscr{V}.
\end{aligned} \tag{2.15}$$

The interpretation is as follows: when player II employs strategy (ρ^*_t,v^*), the strategy (τ^*_t,u^*) maximizes the expected reward of player I over all possible strategies on $\mathscr{S}(t,T)\times\mathscr{U}$; and vice versa, when player I employs strategy (τ^*_t,u^*), the strategy (ρ^*_t,v^*) is optimal for player II over all possible strategies on $\mathscr{S}(t,T)\times\mathscr{V}$. The set of controls and stopping rules (τ^*,ρ^*,u^*,v^*) is called "equilibrium point", or **Nash equilibrium**, for the game. We denote by

$$V^i(t) := \mathbb{E}^{u^*,v^*}[R^i_t(\tau^*_t,\rho^*_t,u^*,v^*)|\mathscr{F}_t], \tag{2.16}$$

the value process of the game for each player $i=1,2$.

In subsections 2.1-2.2 and subsection 2.3, we shall consider two games, which differ in the forms of the rewards R^1 and R^2.

Game 2.1

$$\begin{aligned}
R^1_t(\tau_t,\rho_t,u,v) = R^1_t(\tau_t,u,v) &:= \int_t^{\tau_t} h_1(s,X,u_s,v_s)ds + L_1(\tau_t)\mathbb{1}_{\{\tau_t<T\}} + \xi_1\mathbb{1}_{\{\tau_t=T\}};\\
R^2_t(\tau_t,\rho_t,u,v) = R^2_t(\rho_t,u,v) &:= \int_t^{\rho_t} h_2(s,X,u_s,v_s)ds + L_2(\rho_t)\mathbb{1}_{\{\rho_t<T\}} + \xi_2\mathbb{1}_{\{\rho_t=T\}}.
\end{aligned} \tag{2.17}$$

Game 2.2

$$\begin{aligned}
&R^1_t(\tau_t,\rho_t,u,v)\\
&:= \int_t^{\tau_t\wedge\rho_t} h_1(s,X,u_s,v_s)ds + L_1(\tau_t)\mathbb{1}_{\{\tau_t\leq\rho_t<T\}} + U_1(\rho_t)\mathbb{1}_{\{\rho_t<\tau_t\}} + \xi_1\mathbb{1}_{\{\tau_t\wedge\rho_t=T\}};\\
&R^2_t(\tau_t,\rho_t,u,v)\\
&:= \int_t^{\tau_t\wedge\rho_t} h_2(s,X,u_s,v_s)ds + L_2(\rho_t)\mathbb{1}_{\{\rho_t\leq\tau_t<T\}} + U_2(\tau_t)\mathbb{1}_{\{\tau_t<\rho_t\}} + \xi_2\mathbb{1}_{\{\tau_t\wedge\rho_t=T\}}.
\end{aligned} \tag{2.18}$$

The rewards from both games are the sums of cumulative rewards at rates $h = (h_1, h_2)'$, early exercise rewards $L = (L_1, L_2)'$ and $U = (U_1, U_2)'$, and terminal rewards $\xi = (\xi_1, \xi_2)'$. Here and throughout this paper the notation M' means transpose of some matrix M. The cumulative reward rates h_1 and $h_2 : [0,T] \times \Omega \times \mathbb{A}_1 \times \mathbb{A}_2 \to \mathbb{R}$, $(t, \omega, \mu(t,\omega), \nu(t,\omega)) \mapsto h_i(t, \omega, \mu(t,\omega), \nu(t,\omega))$, $i = 1, 2$, are predictable processes in t, non-anticipative functionals in $X(\cdot)$, and measurable functions in $\mu(t,\omega)$ and $\nu(t,\omega)$. The early exercise rewards $L : [0,T] \times \Omega \to \mathbb{R}^2$, $(t,\omega) \mapsto L(t,\omega) =: L(t)$, and $U : [0,T] \times \Omega \to \mathbb{R}^2$, $(t,\omega) \mapsto U(t,\omega) =: U(t)$ are both $\{\mathscr{F}_t\}_{0\le t\le T}$-adapted continuous processes. The terminal reward $\xi = (\xi_1, \xi_2)'$ is a pair of real-valued $\mathscr{F}_T$-measurable random variables. In the Markovian case, the rewards take the form $h(t, X, u_t, v_t) = h(t, X(t), \mu(t, X(t)), \nu(t, X(t)))$, $L(t) = \bar{L}(t, X(t))$, $U(t) = \bar{U}(t, X(t))$, and $\xi = \bar{\xi}(X(T))$, for all $0 \le t \le T$ and some deterministic measurable functions $\bar{L} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $\bar{U} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, and $\bar{\xi} : \mathbb{R}^d \to \mathbb{R}^2$.

Assumption 2.2 *(1) The early exercise reward processes L and U are continuous, progressively measurable, and $L(T) \le \xi$ holds a.e. on Ω.*
(2) There exist some constants $p \ge 1$ and $C_{rwd} > 0$, such that

$$|h(t, \omega, \mu(t,\omega), \nu(t,\omega))| + |L(t,\omega)| + |U(t,\omega)| + |\xi(\omega)| \le C_{rwd} \left(1 + \sup_{0\le s\le t} |\omega(s)|^{2p}\right) \tag{2.19}$$

holds for a.e. (t,ω) in $[0,T]\times\Omega$, and for all admissible controls $u_t = \mu(t,\omega)$ and $v_t = \nu(t,\omega)$.

From the rewards and the coefficients of the state process, we define the Hamiltonians associated with our games as

$$\begin{aligned} &H_1(t, \omega, z_1, u_t, v_t) = H_1(t, \omega, z_1, \mu(t,\omega), \nu(t,\omega)) \\ :=&z_1 \sigma^{-1}(t,\omega) f(t, \omega, \mu(t,\omega), \nu(t,\omega)) + h_1(t, \omega, \mu(t,\omega), \nu(t,\omega)), \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &H_2(t, \omega, z_2, u_t, v_t) = H_2(t, \omega, z_2, \mu(t,\omega), \nu(t,\omega)) \\ :=&z_2 \sigma^{-1}(t,\omega) f(t, \omega, \mu(t,\omega), \nu(t,\omega)) + h_2(t, \omega, \mu(t,\omega), \nu(t,\omega)), \end{aligned} \tag{2.21}$$

for $0 \le t \le T$, $\omega \in \Omega$, z_1 and z_2 in $\mathbb{R}^d$, and for all admissible controls $u_t = \mu(t,\omega)$ and $v_t = \nu(t,\omega)$. From Assumption 2.1 (3), the Hamiltonians are Lipschitz functions in z_1 and z_2, uniformly over all $0 \le t \le T$, $\omega \in \Omega$, and all admissible controls $u_t = \mu(t,\omega)$ and $v_t = \nu(t,\omega)$.

Assumption 2.3 *(Isaacs condition) There exist admissible controls $u_t^* = \mu^*(t,\omega)$ in $\mathscr{U}$ and $v_t^* = \nu^*(t,\omega)$ in $\mathscr{V}$, such that*

$$\begin{aligned} H_1(t, \omega, z_1, \mu^*(t,\omega), \nu^*(t,\omega)) &\ge H_1(t, \omega, z_1, \mu(t,\omega), \nu^*(t,\omega)), \\ H_2(t, \omega, z_2, \mu^*(t,\omega), \nu^*(t,\omega)) &\ge H_2(t, \omega, z_2, \mu^*(t,\omega), \nu(t,\omega)), \end{aligned} \tag{2.22}$$

hold for all $0 \le t \le T$, $\omega \in \Omega$, $(z_1, z_2) \in \mathbb{R}^{2\times d}$, and for all admissible controls $u_t = \mu(t,\omega)$ and $v_t = \nu(t,\omega)$.

The Isaacs conditions on the Hamiltonians are "local" optimality conditions, formulated in terms of every point (t, z_1, z_2) in Euclidean space and every path ω in the function space Ω. Theorem 2.1 takes the local conditions on the Hamiltonians and transforms them into "global" optimization statements involving each higher-dimensional object, such as stopping time, stochastic process, etc., taking values in Euclidean space and defined over the probability space. This possibility is afforded by the continuous-time setting, in contrast to some discrete-time optimization problems where local maximization need not lead to global maximization.

When linking value processes of the games to the solutions to BSDEs, we shall discuss the solutions in the following spaces $\mathbb{M}^2(m; 0, T)$ and $\mathbb{L}^2(m \times d; 0, T)$ of processes, defined as

$$\mathbb{M}^k(m; t, T) := \left\{ m\text{-dimensional predictable process } \phi(\cdot) \text{ s.t. } \mathbb{E}\left[\sup_{[t,T]} \phi_s^2\right] \leq \infty \right\}, \tag{2.23}$$

and

$$\mathbb{L}^k(m \times d; t, T) := \left\{ m \times d\text{-dimensional predictable process } \phi(\cdot) \text{ s.t. } \mathbb{E}\left[\int_t^T \phi_s^2 dt\right] \leq \infty \right\}, \tag{2.24}$$

for $k = 1, 2$, and $0 \leq t \leq T$.

2.1 The duality between Game and BSDE

This subsection studies Game 2.1 where a player's time to quit is determined by his own decision. We shall demonstrate that the solution to a two-dimensional BSDE with reflecting barrier provides the two players' value processes. The optimal stopping rules will be derived from the reflecting conditions on the BSDE. The optimal controls will come from the Isaacs condition, namely, Assumption 2.3 on the Hamiltonians, which play here the role of the driver of the corresponding BSDE.

The solution to the following system of BSDEs

$$\begin{cases} Y_1^{u,v}(t) = \xi_1 + \int_t^T H_1(s, X, Z_1^{u,v}(s), u_s, v_s) ds - \int_t^T Z_1^{u,v}(s) dB_s + K_1^{u,v}(T) - K_1^{u,v}(t), \\ \qquad Y_1^{u,v}(t) \geq L_1(t),\ 0 \leq t \leq T;\ \int_0^T (Y_1^{u,v}(t) - L_1(t)) dK_1^{u,v}(t) = 0; \\ Y_2^{u,v}(t) = \xi_2 + \int_t^T H_2(s, X, Z_2^{u,v}(s), u_s, v_s) ds - \int_t^T Z_2^{u,v}(s) dB_s + K_2^{u,v}(T) - K_2^{u,v}(t), \\ \qquad Y_2^{u,v}(t) \geq L_2(t),\ 0 \leq t \leq T;\ \int_0^T (Y_2^{u,v}(t) - L_2(t)) dK_2^{u,v}(t) = 0, \end{cases} \tag{2.25}$$

provides the players' value processes in Game 2.1, with the proper choice of controls $u = u^*$ and $v = v^*$ mandated by Isaacs condition. From now on, a BSDE with reflecting barrier in the form of (2.25) will be denoted as $(T, \xi, H(u,v), L)$ for short. The solution to this BSDE is a triple of processes $(Y^{u,v}, Z^{u,v}, K^{u,v})$, satisfying $Y^{u,v}(\cdot) \in \mathbb{M}^2(2; 0, T)$, $Z^{u,v}(\cdot) \in \mathbb{L}^2(2 \times d; 0, T)$, and $K^{u,v}(\cdot) = (K_1^{u,v}(\cdot), K_2^{u,v}(\cdot))'$ a pair of continuous increasing processes in $\mathbb{M}^2(2; 0, T)$.

We focus on the game aspect in this section, making use of such results as existence of solutions to BSDEs, one-dimensional comparison theorems, and continuous dependence properties, to be proved in section 3 and section 4. The proofs of these results will not rely on developments in this section.

Theorem 2.1 *Let $(Y^{u,v}, Z^{u,v}, K^{u,v})$ solve the BSDE (2.25) with parameters $(T, \xi, H(u,v), L)$. Define the stopping rules*

$$\tau_t^*(y; r) := \inf\{s \in [t, r] : y(s) \le L_1(s)\} \wedge r, \tag{2.26}$$

and

$$\rho_t^*(y; r) = \inf\{s \in [t, r] : y(s) \le L_2(s)\} \wedge r, \tag{2.27}$$

for $y \in C[0,T]$ and $r \in [t, T]$. Consider the stopping times

$$\tau_t(u,v) := \tau_t^*\left(Y_1^{u,v}(\cdot); T\right) \quad \text{and} \quad \rho_t(u,v) := \rho_t^*\left(Y_2^{u,v}(\cdot); T\right), \tag{2.28}$$

and suppose that the controls $u^* \in \mathscr{U}$ and $v^* \in \mathscr{V}$ satisfy the Isaacs condition, Assumption 2.3. Then the quadruple $(\tau(u^*, v^*), \rho(u^*, v^*), u^*, v^*)$ is a Nash equilibrium for Game 2.1, and we have $V^i(t) = Y_i^{u^*,v^*}(t)$, $i = 1, 2$.

To prove Theorem 2.1, we shall need the following result.

Lemma 2.1 *For $i = 1, 2$, the process*

$$M_i^{u,v}(\cdot) := \int_t^{\cdot} Z_i^{u,v}(s) dB_s^{u,v} = Y_i^{u,v}(\cdot) - Y_i^{u,v}(t) + \int_t^{\cdot} h_i(s, X, u_s, v_s) ds + K_i^{u,v}(\cdot) - K_i^{u,v}(t) \tag{2.29}$$

is a $\mathbb{P}^{u,v}$-martingale.

Proof. To show that $M_i^{u,v}(\cdot)$ is a $\mathbb{P}^{u,v}$-martingale, it suffices to show that $M_i^{u,v}(\cdot)$ is of class $\mathscr{DL}$, meaning that

$$\lim_{c \to \infty} \sup_{\tau \in \mathscr{S} t, T} \mathbb{E}^{u,v}\left[|M_i^{u,v}(\tau)| \, \mathbb{1}_{\{|M_i^{u,v}(\tau)| > c\}} \middle| \mathscr{F}_t \right] = 0. \tag{2.30}$$

For the fixed $t \in [0, T]$, denote

$$\theta(s, u_s, v_s) := \sigma^{-1}(s, X) f(s, X, u_s, v_s), \ t \le s \le T. \tag{2.31}$$

For any $\tau \in \mathscr{S}(t, T)$, from the change of measure (2.9) and the Bayes rule,

$$\begin{aligned}
&\mathbb{E}^{u,v}\left[|M_i^{u,v}(\tau)| \, \mathbb{1}_{\{|M_i^{u,v}(\tau)| > c\}} \middle| \mathscr{F}_t \right] \\
=&\mathbb{E}\left[\exp\left\{ \int_t^\tau \theta(s, u_s, v_s) dB_s - \frac{1}{2} \int_t^\tau |\theta(s, u_s, v_s)|^2 ds \right\} |M_i^{u,v}(\tau)| \, \mathbb{1}_{\{|M_i^{u,v}(\tau)| > c\}} \middle| \mathscr{F}_t \right] \\
\le&\mathbb{E}\left[\sup_{t \le s \le T} \exp\left\{ \int_t^s 2\theta(r, u_r, v_r) dB_r - \frac{1}{2} \int_t^s 2|\theta(r, u_r, v_r)|^2 dr \right\} \middle| \mathscr{F}_t \right]^{1/2} \\
&\cdot \mathbb{E}\left[\sup_{t \le s \le T} |M_i^{u,v}(s)|^2 \, \mathbb{1}_{\{\sup_{t \le s \le T} |M_i^{u,v}(s)|^2 > c^2\}} \middle| \mathscr{F}_t \right]^{1/2}.
\end{aligned} \tag{2.32}$$

From the expression (2.29) and Assumption 2.2 (2), there exists some constant C_0, such that

$$\sup_{t\le s\le T} |M_i^{u,v}(s)|^2 \le C_0 \left(1 + \sup_{t\le s\le T} |Y_i^{u,v}(s)|^2 + \sup_{t\le s\le T} |X(s)|^{2p} + |K_i^{u,v}(T)|^2\right). \tag{2.33}$$

From the definition of the solutions to reflected BSDEs, as in section 3, we know that

$$\mathbb{E}\left[\sup_{t\le s\le T} (Y_i^{u,v}(s))^2 + |K_i^{u,v}(T)|^2\right] < \infty \tag{2.34}$$

holds. Since (X, B) is a solution to the stochastic functional equation (2.2), there exists (cf. page 306 of Karatzas and Shreve (1988) [28]) a constant C_1 such that

$$\mathbb{E}\left[\sup_{t\le s\le T} |X(s)|^{2p}\right] \le C_1 \left(1 + |X(0)|^{2p}\right) < \infty. \tag{2.35}$$

We then apply the dominated convergence theorem to the last conditional expectations in (2.32) to get

$$\mathbb{E}\left[\sup_{t\le s\le T} |M_i^{u,v}(s)|^2 \mathbb{1}_{\{\sup_{t\le s\le T} |M_i^{u,v}(s)|^2 > c^2\}} \middle| \mathscr{F}_t\right] \to 0, \tag{2.36}$$

as $c \to 0$. It remains to show that

$$\mathbb{E}\left[\sup_{t\le s\le T} \exp\left\{\int_t^s 2\theta(r, u_r, v_r)dB_r - \frac{1}{2}\int_t^s 2|\theta(r, u_r, v_r)|^2 dr\right\}\right] < \infty. \tag{2.37}$$

Because $|\theta(s, u_s, v_s)|$ is bounded by the constant A, from Assumption 2.1 (3) and identity (2.31), we know that the process

$$\exp\left\{\int_t^{\cdot} 2\theta(s, u_s, v_s)dB_s - \frac{1}{2}\int_t^{\cdot} 2|\theta(s, u_s, v_s)|^2 ds\right\} \tag{2.38}$$

is a.e. bounded by the constant $e^{A^2 T}$ times the exponential $\mathbb{P}$-martingale

$$Q(\cdot) := \exp\left\{\int_t^{\cdot} 2\theta(s, u_s, v_s)dB_s - \frac{1}{2}\int_t^{\cdot} 4|\theta(s, X, u_s, v_s)|^2 ds\right\} \tag{2.39}$$

on $[0, T]$ with quadratic variation process

$$\langle Q\rangle(\cdot) = 4\int_t^{\cdot} Q^2(s)|\theta(s, u_s, v_s)|^2 ds. \tag{2.40}$$

But

$$\begin{aligned} &Q^2(\cdot)|\theta(\cdot, u_\cdot, v_\cdot)|^2 \\ \le &A^2 e^{4A^2 T} \exp\left\{\int_t^{\cdot} 4\theta(s, u_s, v_s)dB_s - \frac{1}{2}\int_t^{\cdot} 16|\theta(s, u_s, v_s)|^2 ds\right\}. \end{aligned} \tag{2.41}$$

By the Burkholder-Davis-Gundy inequalities and inequality (2.41), there exists a constant C, such that $\mathbb{E}\left[\sup_{t\leq s\leq T} Q(s)\right]$ is dominated by

$$\begin{aligned}
&2CAe^{2A^2T}\mathbb{E}\left[\left(\int_t^T \exp\left\{\int_t^s 4\theta(r,u_r,v_r)dB_r - \frac{1}{2}\int_t^s 16|\theta(r,u_r,v_r)|^2dr\right\}ds\right)^{1/2}\right]\\
\leq&2CAe^{2A^2T}\left(\int_t^T \mathbb{E}\left[\exp\left\{\int_t^s 4\theta(r,u_r,v_r)dB_r - \frac{1}{2}\int_t^s 16|\theta(r,u_r,v_r)|^2dr\right\}\right]ds\right)^{1/2}\\
=&2CAe^{2A^2T}(T-t)^{1/2}.
\end{aligned}$$

This proves (2.37), whereas the expressions (2.32), (2.36) and (2.37) together lead to (2.30). □

Proof of Theorem 2.1. Let $(Y^{u,v}, Z^{u,v}, K^{u,v})$ solve BSDE (2.25) with parameters $(T, \xi, H(u,v), L)$. Taking a stopping rule $\tau_t \in \mathscr{S}(t,T)$, and integrating $dY_1^{u,v}$ from t to τ_t, we obtain

$$\begin{aligned}
Y_1^{u,v}(t) =&Y_1^{u,v}(\tau_t) + \int_t^{\tau_t} H_1(s,X,Z_1^{u,v}(s),u_s,v_s)ds - \int_t^{\tau_t} Z_1^{u,v}(s)dB_s + K_1^{u,v}(\tau_t) - K_1^{u,v}(t)\\
=&Y_1^{u,v}(\tau_t) + \int_t^{\tau_t} h_1(s,X,u_s,v_s)ds - \int_t^{\tau_t} Z_1^{u,v}(s)dB_s^{u,v} + K_1^{u,v}(\tau_t) - K_1^{u,v}(t).
\end{aligned} \tag{2.42}$$

Taking conditional expectation $\mathbb{E}^{u,v}[\cdot|\mathscr{F}_t]$, and using the comparisons $Y_1^{u,v}(\cdot) \geq L_1(\cdot)$, $Y_1^{u,v}(T) = \xi_1$, as well as the fact that $K_1^{u,v}(\cdot)$ is an increasing process, we obtain

$$\begin{aligned}
Y_1^{u,v}(t) =&\mathbb{E}^{u,v}\left[Y_1^{u,v}(\tau_t) + \int_t^{\tau_t} h_1(s,X,u_s,v_s)ds + K_1^{u,v}(\tau_t) - K_1^{u,v}(t)\middle|\mathscr{F}_t\right]\\
\geq&\mathbb{E}^{u,v}\left[L_1(\tau_t)\mathbb{1}_{\{\tau_t\wedge T_1^n<T\}} + \xi_1\mathbb{1}_{\{\tau_t\}} + \int_t^{\tau_t} h_1(s,X,u_s,v_s)ds\middle|\mathscr{F}_t\right].
\end{aligned} \tag{2.43}$$

According to the reflecting condition in BSDE (2.25), $K_1^{u,v}(\cdot)$ is flat on $\{(\omega,t)\in(\Omega\times[0,T]): Y_1^{u,v}(t)\neq L_1(t)\}$; from the continuity of $K_1^{u,v}(\cdot)$, we see that $K_1^{u,v}(\tau_t(u,v)) = K_1^{u,v}(t)$. On $\{\tau_t(u,v)<T\}$, $Y_1^{u,v}(\tau_t(u,v)) = L_1(\tau_t(u,v))$; on $\{\tau_t(u,v)=T\}$, $Y_1^{u,v}(\tau_t(u,v)) = \xi_1$. Then,

$$\begin{aligned}
&Y_1^{u,v}(t)\\
=&\mathbb{E}^{u,v}\left[Y_1^{u,v}(\tau_t(u,v)) + \int_t^{\tau_t(u,v)} h_1(s,X,u_s,v_s)ds\middle|\mathscr{F}_t\right]\\
=&\mathbb{E}^{u,v}\left[L_1(\tau_t(u,v))\mathbb{1}_{\{\tau_t(u,v)<T\}} + \xi_1\mathbb{1}_{\{\tau_t(u,v)=T\}} + \int_t^{\tau_t(u,v)} h_1(s,X,u_s,v_s)ds\middle|\mathscr{F}_t\right].
\end{aligned} \tag{2.44}$$

The expressions (2.44) and (2.43) mean that

$$Y_1^{u,v}(t) = \mathbb{E}^{u,v}[R_t^1(\tau_t(u,v),\rho_t,u,v)|\mathscr{F}_t] \geq \mathbb{E}^{u,v}[R_t^1(\tau_t,\rho_t,u,v)|\mathscr{F}_t] \tag{2.45}$$

holds for all $\rho_t \in \mathscr{S}(t,T)$ and for all $\tau_t \in \mathscr{S}(t,T)$.

To derive the optimality of the controls (u^*, v^*) from the Isaacs condition, Assumption 2.3, an application of the comparison theorem (Theorem 3.2 and 4.3) to the first component of BSDE (2.25) gives $Y_1^{u^*,v^*}(\cdot) \geq Y_1^{u,v^*}(\cdot)$ a.e. on $[0,T]\times\Omega$. From the identity in (2.45), we have

$$\begin{aligned}&\mathbb{E}^{u,v^*}[R_t^1(\tau_t(u^*,v^*),\rho_t(u^*,v^*),u^*,v^*)|\mathscr{F}_t] = Y_1^{u^*,v^*}(t)\\ \geq &Y_1^{u,v^*}(t) = \mathbb{E}^{u,v^*}[R_t^1(\tau_t(u,v^*),\rho_t(u,v^*),u,v^*)|\mathscr{F}_t];\end{aligned} \tag{2.46}$$

and in conjunction with (2.45), for all $\tau_t \in \mathscr{S}(t,T)$, this gives

$$\begin{aligned}&\mathbb{E}^{u^*,v^*}[R_t^1(\tau_t(u^*,v^*),\rho_t(u^*,v^*),u^*,v^*)|\mathscr{F}_t]\\ \geq&\mathbb{E}^{u,v^*}[R_t^1(\tau_t(u,v^*),\rho_t(u,v^*),u,v^*)|\mathscr{F}_t]\\ \geq&\mathbb{E}^{u,v^*}[R_t^1(\tau_t,\rho_t(u,v^*),u,v^*)|\mathscr{F}_t].\end{aligned} \tag{2.47}$$

The above arguments proceed with arbitrary stopping times $\rho_t \in \mathscr{S}(t,T)$, because player II's stopping time ρ_t does not enter player I's reward.

By symmetry between the two players,

$$Y_2^{u^*,v^*} = \mathbb{E}^{u^*,v^*}[R_t^2(\tau_t(u^*,v^*),\rho_t(u^*,v^*),u^*,v^*)|\mathscr{F}_t], \tag{2.48}$$

and

$$\mathbb{E}^{u^*,v^*}[R_t^2(\tau_t(u^*,v^*),\rho_t(u^*,v^*),u^*,v^*)|\mathscr{F}_t] \geq \mathbb{E}^{u^*,v}[R_t^2(\tau_t(u^*,v^*),\rho_t,u^*,v)|\mathscr{F}_t]. \tag{2.49}$$

Combining (2.46), (2.47), (2.48) and (2.49), we see that the quadruple (τ^*,ρ^*,u^*,v^*) is a Nash equilibrium and their value processes $V^1(\cdot)$ and $V^2(\cdot)$ are identified with the solution to a BSDE with reflecting barrier with parameters $(T,\xi,H(u^*,v^*),L)$, as in (2.25). The optimal controls (u^*,v^*) are chosen according to the Isaacs condition, Assumption 2.3. Both players stop respectively according to the pair of rules (τ_t^*,ρ_t^*), as soon as their conditional expected rewards $V^1(\cdot)$ and $V^2(\cdot)$ hit the early stopping rewards $L_1(\cdot)$ and $L_2(\cdot)$ for the first time. □

Remark 2.1 *If the deterministic time T is replaced by a bounded $\{\mathscr{F}_t\}_{0\leq t\leq T}$-stopping time, it technically does not make any difference to results in this subsection.*

2.2 Controls observing volatility

This subsection discusses whether the inclusion of instantaneous volatilities of the value processes into the controls will expand the admissible control sets.

For the rewards considered in this paper, when using control u and v, the $\mathbb{P}^{u,v}$-conditional expected rewards are $\mathbb{P}^{u,v}$-Brownian semimartingales with respect to the filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$, having the decompositions

$$\begin{aligned}\mathbb{E}^{u,v}[R^1_t(\tau,\rho,u,v)|\mathscr{F}_t] =& A^{u,v}_1(t) + M^{u,v}_1(t) = A^{u,v}_1(t) + \int_0^t Z^{u,v}_1(s)dB^{u,v}_1(s);\\ \mathbb{E}^{u,v}[R^2_t(\tau,\rho,u,v)|\mathscr{F}_t] =& A^{u,v}_2(t) + M^{u,v}_2(t) = A^{u,v}_2(t) + \int_0^t Z^{u,v}_2(s)dB^{u,v}_2(s).\end{aligned} \tag{2.50}$$

The processes $A^1(\cdot)$ and $A^2(\cdot)$ are adapted and have finite variation. The processes $M^1(\cdot)$ and $M^2(\cdot)$ are $\mathbb{P}^{u,v}$-local martingales with respect to $\{\mathscr{F}_t\}_{0\le t\le T}$. The predictable, square-integrable processes $Z^{u,v}_1(\cdot)$ and $Z^{u,v}_2(\cdot)$ from martingale representation are called instantaneous volatility processes, the very integrand processes of the stochastic integrals in the BSDE (2.25). Because they naturally show up in the BSDEs solved by value process of the game, we may include the instantaneous volatilities $Z^{u,v}_1(\cdot)$ and $Z^{u,v}_2(\cdot)$ as arguments of the controls u and v, in the hope of making more informed decisions. Going one step further, in the case of risk-sensitive controls initiated by Whittle, Bensoussan and coworkers, among others, for example Bensoussan, Frehse and Nagai (1998) [4], the players are sensitive not only to the expectations, but also to the variances of their rewards; we emphasize sensitivity to volatilities by including them as arguments of the controls. El Karoui and Hamadène (2003) identified in [11] risk-sensitive controls to BSDEs with quadratic growth in $Z^{u,v}_1(\cdot)$ and $Z^{u,v}_2(\cdot)$, which made the problem very tractable. Their value processes are different from the risk-indifferent case only up to an exponential transformation.

Apply the controls $u_t = \mu(t, X, Z_1(t), Z_2(t))$ and $v_t = \nu(t, X, Z_1(t), Z_2(t))$, for some deterministic measurable functionals $\mu : [0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{A}_1$ and $\nu : [0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{A}_2$, and for some $\{\mathscr{F}_t\}_{0\le t\le T}$-predictable processes $Z_1(\cdot)$ and $Z_2(\cdot)$. If the resulting instantaneous volatilities $Z^{u,v}_1(\cdot)$ and $Z^{u,v}_2(\cdot)$ in the semimartingale decomposition (2.50) coincide with the arguments $Z_1(\cdot)$ and $Z_2(\cdot)$ of the functionals μ and ν, then u and v are said to be *a pair of closed loop controls that observe the instantaneous volatilities.*

Apply the controls $u_t = \mu(t, X(t), Z_1(t), Z_2(t))$ and $v_t = \nu(t, X(t), Z_1(t), Z_2(t))$, for some deterministic measurable functions $\mu : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{A}_1$ and $\nu : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{A}_2$, and for some $\{\mathscr{F}_t\}_{0\le t\le T}$-predictable processes $Z_1(\cdot)$ and $Z_2(\cdot)$. If the resulting instantaneous volatilities $Z^{u,v}_1(\cdot)$ and $Z^{u,v}_2(\cdot)$ in the semimartingale decomposition (2.50) coincide with the arguments $Z_1(\cdot)$ and $Z_2(\cdot)$ of the functions μ and ν, then u and v are said to be *a pair of Markovian controls that observe the instantaneous volatilities.* This is the case about which we are going to have more to say.

The Hamiltonians in this case become

$$\begin{aligned}&H_1(t,\omega(t),z_1,(\mu,\nu)(t,\omega(t),z_1,z_2))\\ =&z_1\sigma^{-1}(t,\omega(t))f(t,\omega(t),(\mu,\nu)(t,\omega(t),z_1,z_2)) + h_1(t,\omega(t),(\mu,\nu)(t,\omega(t),z_1,z_2))\end{aligned} \tag{2.51}$$

and

$$\begin{aligned}&H_2(t,\omega(t),z_2,(\mu,\nu)(t,\omega(t),z_1,z_2))\\ =&z_2\sigma^{-1}(t,\omega(t))f(t,\omega(t),(\mu,\nu)(t,X(t),z_1,z_2)) + h_2(t,\omega(t),(\mu,\nu)(t,\omega(t),z_1,z_2)),\end{aligned} \tag{2.52}$$

for $0 \le t \le T$, $\omega \in \Omega$, z_1 and z_2 in $\mathbb{R}^d$, and $\mathbb{A}_1 \times \mathbb{A}_2$-valued measurable functions (μ, ν). From Assumption 2.1 (3) and Assumption 2.2 (2), the Hamiltonians are liner in z_1 and z_2, and polynomial in $\sup_{0 \le s \le t} |\omega(s)|$. To be more specific, we have

$$|H_i(t, \omega(t), z_1, z_2, (\mu, \nu)(t, \omega(t), z_1, z_2))| \le A|z_i| + C_{\text{rwd}} \left(1 + \sup_{0 \le s \le t} |\omega(s)|^{2p}\right), \tag{2.53}$$

for $i = 1, 2$, all $0 \le t \le T$, $\omega \in \Omega$, z_1 and z_2 in $\mathbb{R}^d$, and $\mathbb{A}_1 \times \mathbb{A}_2$-valued measurable functions (μ, ν). The growth rates of the Hamiltonians (2.51) and (2.52) satisfy Assumption 4.1 (2) for the driver of the BSDE (4.2). With all other assumptions on the coefficients also satisfied, by Theorem 4.2, there exists a solution $(Y^{\mu,\nu}, Z^{\mu,\nu}, K^{\mu,\nu})$ to the following equation

$$\begin{cases} Y_1^{\mu,\nu}(t) = \xi_1 + \int_t^T H_1(s, X(s), Z_1^{\mu,\nu}(s), (\mu, \nu)(s, X(s), Z_1^{\mu,\nu}(s), Z_2^{\mu,\nu}(s)))ds \\ \qquad - \int_t^T Z_1^{\mu,\nu}(s) dB_s + K_1^{\mu,\nu}(T) - K_1^{\mu,\nu}(t), \\ \qquad Y_1^{\mu,\nu}(t) \ge L_1(t),\ 0 \le t \le T;\ \int_0^T (Y_1^{\mu,\nu}(t) - L_1(t)) dK_1^{\mu,\nu}(t) = 0; \\ Y_2^{\mu,\nu}(t) = \xi_2 + \int_t^T H_2(s, X(s), Z_2^{\mu,\nu}(s), (\mu, \nu)(s, X(s), Z_1^{\mu,\nu}(s), Z_2^{\mu,\nu}(s)))ds \\ \qquad - \int_t^T Z_2^{\mu,\nu}(s) dB_s + K_2^{\mu,\nu}(T) - K_2^{\mu,\nu}(t), \\ \qquad Y_2^{\mu,\nu}(t) \ge L_2(t),\ 0 \le t \le T;\ \int_0^T (Y_2^{\mu,\nu}(t) - L_2(t)) dK_2^{\mu,\nu}(t) = 0. \end{cases} \tag{2.54}$$

Assumption 2.4 *(Isaacs condition) There exist deterministic functions $\mu^* : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{A}_1$ and $\nu^* : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{A}_2$, such that*

$$\begin{aligned} H_1(t, x, z_1, (\mu^*, \nu^*)(t, x, z_1, z_2)) &= \sup_{\bar{z}_1, \bar{z}_2 \in \mathbb{R}^d} H_1(t, x, z_1, (\mu, \nu^*)(t, x, \bar{z}_1, \bar{z}_2)); \\ H_2(t, x, z_2, (\mu^*, \nu^*)(t, x, z_1, z_2)) &= \sup_{\bar{z}_1, \bar{z}_2 \in \mathbb{R}^d} H_2(t, x, z_2, (\mu^*, \nu)(t, x, \bar{z}_1, \bar{z}_2)), \end{aligned} \tag{2.55}$$

for all $0 \le t \le T$, x, z_1 and z_2 in $\mathbb{R}^d$, and all $\mu : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{A}_1$ and $\nu : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{A}_2$.

Associated with the coefficients f and σ of the state process $X(\cdot)$ and with the rewards h, $L(\cdot)$ and ξ, the admissible set $\mathscr{U} \times \mathscr{V} = \{(u, v)\}$ of Markovian controls that observe volatilities are defined as the collection of all

$$(u_t, v_t) = (\mu, \nu)(t, X(t), Z_1^{\mu,\nu}(t), Z_2^{\mu,\nu}(t)), \tag{2.56}$$

for measurable functions $\mu : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{A}_1$ and $\nu : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{A}_2$. In particular,

$$(u_t^*, v_t^*) = (\mu^*, \nu^*)(t, X(t), Z_1^{\mu^*,\nu^*}(t), Z_2^{\mu^*,\nu^*}(t)), \tag{2.57}$$

$$(u_t, v_t^*) = (\mu, \nu^*)(t, X(t), Z_1^{\mu,\nu^*}(t), Z_2^{\mu,\nu^*}(t)), \tag{2.58}$$

and

$$(u_t^*, v_t) = (\mu^*, \nu)(t, X(t), Z_1^{\mu^*,\nu}(t), Z_2^{\mu^*,\nu}(t)). \tag{2.59}$$

Assumption 2.4 implies the Isaacs condition of Assumption 2.3. Then we reach the same statements as in Theorem 2.1, the only difference being that $(Y^{u,v}, Z^{u,v}, K^{u,v})$ is now replaced by $(Y^{\mu,\nu}, Z^{\mu,\nu}, K^{\mu,\nu})$, and the BSDE (2.25) is replaced by the BSDE (2.54).

In fact, by Theorem 4.1, there exist deterministic measurable mappings $\beta_1^{\mu,\nu}$ and $\beta_2^{\mu,\nu}$: $[0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, such that $Z_1^{\mu,\nu}(t) = \beta_1^{\mu,\nu}(t, X(t))$, and $Z_2^{\mu,\nu}(t) = \beta_2^{\mu,\nu}(t, X(t))$ hold for all $0 \le t \le T$. Hence (2.56) becomes

$$(u_t, v_t) = (\mu, \nu)(t, X(t), \beta_1^{\mu,\nu}(t, X(t)), \beta_2^{\mu,\nu}(t, X(t))), \tag{2.60}$$

a pair of Markovian controls.

2.3 Rewards Terminated by Both Players

In this subsection, we shall discuss the existence of equilibrium for Game 2.2 when rewards can be terminated by both players, in a sense weaker than (2.15). The existence of an equilibrium as in (2.15) is still not quite clear in this case. What also remains open is to write down a dual BSDE associated with this game, the solution to which can then be shown to exist.

By entering Game 2.2 at time t, player I receives reward

$$R_t^1(\tau, \rho, u, v) = \int_t^{\tau\wedge\rho} h_1(s, X, u_s, v_s)ds + \begin{cases} L_1(\tau), \text{ if player 1 stops first;} \\ U_1(\rho), \text{ if player 2 stops first;} \\ \xi_1, \text{ if neither stops before time } T \end{cases} \tag{2.61}$$

at the time when the game is terminated, whereas player II receives reward

$$R_t^2(\tau, \rho, u, v) = \int_t^{\tau\wedge\rho} h_2(s, X, u_s, v_s)ds + \begin{cases} U_2(\tau), \text{ if player 1 stops first;} \\ L_2(\rho), \text{ if player 2 stops first;} \\ \xi_2, \text{ if neither stops before time } T \end{cases} \tag{2.62}$$

when the game is terminated.

To prove the existence of equilibrium for Game 2.2, we shall first study the following game.

Game 2.2'

$$\begin{aligned} &\bar{R}_t^1(\tau_t, \rho_t, u, v) \\ &:= \int_t^{\tau_t\wedge\rho_t} h_1(s, X, u_s, v_s)ds + (L_1 \vee U_1)(\rho_t \wedge \tau_t)\mathbb{1}_{\{\tau_t\wedge\rho_t<T\}} + \xi_1\mathbb{1}_{\{\tau_t\wedge\rho_t=T\}}; \\ &\bar{R}_t^2(\tau_t, \rho_t, u, v) \\ &:= \int_t^{\tau_t\wedge\rho_t} h_2(s, X, u_s, v_s)ds + (L_2 \vee U_2)(\rho_t \wedge \tau_t)\mathbb{1}_{\{\tau_t\wedge\rho_t<T\}} + \xi_2\mathbb{1}_{\{\tau_t\wedge\rho_t=T\}}, \end{aligned} \tag{2.63}$$

where

$$(L_i \vee U_i)(t) := \max\{L_i(t), U_i(t)\}, \tag{2.64}$$

for $0 \le t \le T$, *and* $i = 1, 2$.

Lemma 2.2 *Under Assumptions 2.1 - 2.3, there exists an equilibrium* $(\bar{\tau}^*, \bar{\rho}^*, u^*, v^*)$ *for Game 2.2', in the sense of (2.15) with* R^1 *and* R^2 *replaced by* $\bar{R}^1$ *and* $\bar{R}^2$.

Proof. The proof follows the inductive scheme proposed in Karatzas and Sudderth (2006) [29]. Suppose that the controls $u^* \in \mathscr{U}$ and $v^* \in \mathscr{V}$ satisfy the Isaacs condition, Assumption 2.3. Let $\tau_t^0 = \rho_t^0 = T$. Define the stopping rules

$$\bar{\tau}_t(y; r) = \inf\{s \in [t, r]\,|y(s) \le (L_1 \vee U_1)(s)\} \wedge r, \tag{2.65}$$

and

$$\bar{\rho}_t(y; r) = \inf\{s \in [t, r]\,|y(s) \le (L_2 \vee U_2)(s)\} \wedge r, \tag{2.66}$$

for $y \in C[0, T]$ and $r \in [t, T]$. For $n = 0, 1, 2, \cdots$, define the value functions as

$$\bar{V}^1_{n+1}(t) := \sup_{\tau_t \in \mathscr{S}(t,T)} \mathbb{E}^{u^*,v^*}[\bar{R}^1_t(\tau_t, \rho^n_t, u^*, v^*)|\mathscr{F}_t], \tag{2.67}$$

and

$$\bar{V}^2_{n+1}(t) := \sup_{\rho_t \in \mathscr{S}(t,T)} \mathbb{E}^{u^*,v^*}[\bar{R}^2_t(\tau^n_t, \rho_t, u^*, v^*)|\mathscr{F}_t]. \tag{2.68}$$

The stopping times

$$\tau^{n+1}_t := \bar{\tau}_t(\bar{V}^1_{n+1}; \rho^n_t), \tag{2.69}$$

and

$$\rho^{n+1}_t := \bar{\rho}_t(\bar{V}^2_{n+1}; \tau^n_t) \tag{2.70}$$

achieve the suprema in (2.67) and in (2.68), respectively. By applying Theorem 2.1 in dimension one to each individual player, we know that the inequalities

$$\mathbb{E}^{u^*,v^*}[\bar{R}^1_t(\tau^{n+1}_t, \rho^n_t, u^*, v^*)|\mathscr{F}_t] \ge \mathbb{E}^{u,v^*}[\bar{R}^1_t(\tau_t, \rho^n_t, u, v^*)|\mathscr{F}_t],\ \forall \tau_t \in \mathscr{S}(t,T),\ \forall u \in \mathscr{U}, \tag{2.71}$$

and

$$\mathbb{E}^{u^*,v^*}[\bar{R}^2_t(\tau^n_t, \rho^{n+1}_t, u^*, v^*)|\mathscr{F}_t] \ge \mathbb{E}^{u^*,v}[\bar{R}^2_t(\tau^n_t, \rho_t, u^*, v)|\mathscr{F}_t],\ \forall \rho_t \in \mathscr{S}(t,T),\ \forall v \in \mathscr{V} \tag{2.72}$$

hold for a.e. (t, ω) in $[0, T] \times \Omega$.
The comparisons $\tau^1_t \le \tau^0_t$ and $\rho^1_t \le \rho^0_t$ imply that $\bar{V}^1_2(t) \le \bar{V}^1_1(t)$ and $\bar{V}^2_2(t) \le \bar{V}^2_1(t)$. Inductively, we know that $\tau^{n+1}_t \le \tau^n_t$, $\rho^{n+1}_t \le \rho^n_t$, $\bar{V}^1_{n+1}(t) \le \bar{V}^1_n(t)$ and $\bar{V}^2_{n+1}(t) \le \bar{V}^2_n(t)$, for all $0 \le t \le T$, and all $n = 0, 1, 2, \cdots$. The decreasing sequences of the value functions $\{\bar{V}^1_n(t)\}_{n\in\mathbb{N}}$, $\{\bar{V}^2_n(t)\}_{n\in\mathbb{N}}$, and the stopping times $\{\tau^n_t\}_{n\in\mathbb{N}}$, $\{\rho^n_t\}_{n\in\mathbb{N}}$, have limits $\bar{V}^1_*(t)$, $\bar{V}^2_*(t)$, $\bar{\tau}^*_t$ and $\bar{\rho}^*_t$. Furthermore, the stopping times $\bar{\tau}^*_t$ and $\bar{\rho}^*_t$ satisfy

$$\bar{\tau}^*_t = \bar{\tau}_t(\bar{V}^1_*; \bar{\rho}^*_t) \le \bar{\rho}^*_t, \tag{2.73}$$

and

$$\bar{\rho}^*_t = \bar{\rho}_t(\bar{V}^2_*; \bar{\tau}^*_t) \le \bar{\tau}^*_t, \tag{2.74}$$

hence

$$\bar{\tau}_t^* = \bar{\rho}_t^*. \tag{2.75}$$

By the continuity of the early exercise rewards $L(\cdot)$ and $U(\cdot)$, sending $n \to \infty$ in (2.71) and (2.72), we conclude that

$$\mathbb{E}^{u^*,v^*}[\bar{R}_t^1(\bar{\tau}_t, \bar{\rho}_t, u^*, v^*)|\mathscr{F}_t] \geq \mathbb{E}^{u,v^*}[\bar{R}_t^1(\tau_t, \bar{\rho}_t, u, v^*)|\mathscr{F}_t],\ \forall \tau_t \in \mathscr{S}(t,T),\ \forall u \in \mathscr{U}, \tag{2.76}$$

and

$$\mathbb{E}^{u^*,v^*}[\bar{R}_t^2(\bar{\tau}_t, \bar{\rho}_t, u^*, v^*)|\mathscr{F}_t] \geq \mathbb{E}^{u^*,v}[\bar{R}_t^2(\bar{\tau}_t, \rho_t, u^*, v)|\mathscr{F}_t],\ \forall \rho_t \in \mathscr{S}(t,T),\ \forall v \in \mathscr{V}. \tag{2.77}$$

□

Assumption 2.5 *There exist adapted processes* $\gamma_i : [0,T] \times \Omega \to \{0,1\}$, $(t,\omega) \mapsto \gamma_i(t,\omega) =: \gamma_i(t)$, *such that*

$$\gamma_i(t)L_i(t) + (1-\gamma_i(t))U_i(t) = (L_i \vee U_i)(t), \tag{2.78}$$

$i = 1, 2$, *and*

$$\gamma_1(t) + \gamma_2(t) \geq 1 \tag{2.79}$$

hold for all $0 \leq t \leq T$ *and* $\omega \in \Omega$.

Theorem 2.2 *In addition to Assumptions 2.1-2.3, if Assumption 2.5 also holds, then there exists an equilibrium* $(\tau^*, \rho^*, u^*, v^*)$ *for Game 2.2, in the sense that*

$$\begin{aligned}
&\mathbb{E}^{u^*,v^*}[R_t^1(\tau_t^*, \rho_t^*, u^*, v^*)|\mathscr{F}_t] \geq \mathbb{E}^{u,v^*}[R_t^1(\tau_t, \rho_t^*, u, v^*)|\mathscr{F}_t],\ \forall \tau_t \in \mathscr{S}(t, \tau_t^* \wedge \rho_t^*),\ \forall u \in \mathscr{U};\\
&\mathbb{E}^{u^*,v^*}[R_t^2(\tau_t^*, \rho_t^*, u^*, v^*)|\mathscr{F}_t] \geq \mathbb{E}^{u^*,v}[R_t^2(\tau_t^*, \rho_t, u^*, v)|\mathscr{F}_t],\ \forall \rho_t \in \mathscr{S}(t, \tau_t^* \wedge \rho_t^*),\ \forall v \in \mathscr{V}.
\end{aligned} \tag{2.80}$$

Proof. Let $(\bar{\tau}^*, \bar{\rho}^*, u^*, v^*)$ be the equilibrium for Game 2.2', the quadruple specified in Lemma 2.2. Denote by $T_t^* := \bar{\tau}_t^* \wedge \bar{\rho}_t^*$. Define the stopping rules

$$\tau_t^* := \begin{cases} \bar{\tau}_t^* = \bar{\tau}_t(\bar{V}_*^1; T_t^*), & \text{if } \gamma_1(T_t^*) = 1, \text{ or } T_t^* = T; \\ \text{any stopping rule with values in } (T_t^*, T], & \text{if } \gamma_1(T_t^*) = 0, \end{cases} \tag{2.81}$$

and

$$\rho_t^* := \begin{cases} \bar{\rho}_t^* = \bar{\rho}_t(\bar{V}_*^2; T_t^*), & \text{if } \gamma_2(T_t^*) = 1, \text{ or } T_t^* = T; \\ \text{any stopping rule with values in } (T_t^*, T], & \text{if } \gamma_2(T_t^*) = 0. \end{cases} \tag{2.82}$$

By Lemma 2.2, one can verify that the quadruple $(\tau^*, \rho^*, u^*, v^*)$ is an equilibrium for Game 2.2, in the sense of (2.80). Furthermore, $\bar{V}_*^i(t) = V^i(t)$ holds for all $0 \leq t \leq T$, $\omega \in \Omega$, and $i = 1, 2$. □

3 A multi-dimensional reflected BSDE with Lipschitz growth

Starting with this section, we solve multi-dimensional BSDEs with reflecting barriers, the type of BSDEs associated with Game 2.1, and provide two useful properties of the equations: the comparison theorem in dimension one, and the theorem about continuous dependence of the solution on the terminal values. We have postponed the study of the BSDEs up until this point, in order to discuss the Stochastic Game aspects first. The proofs of the results to be stated from now onwards in this paper, do not depend on any earlier arguments or developments.

This section assumes the following Lipschitz and integrability conditions on the parameters of the equations.

Assumption 3.1 *(1) The driver g is a mapping $g : [0,T] \times \mathbb{R}^m \times^{m\times d} \to \mathbb{R}^m$, $(t,y,z) \mapsto g(t,y,z)$. For every fixed $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^{m\times d}$, the process $\{g(t,y,z)\}_{0\le t\le T}$ is $\{\mathscr{F}_t\}_{0\le t\le T}$-predictable. For all $t \in [0,T]$, $g(t,y,z)$ is uniformly Lipschitz in y and z, i.e. there exists a constant $b > 0$, such that*

$$|g(t,y,z) - g(t,\bar{y},\bar{z})| \le b(|y-\bar{y}| + |z-\bar{z}|), \tag{3.1}$$

for all $t \in [0,T]$, $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^{m\times d}$. Furthermore,

$$\int_0^T g(t,0,0)^2 dt < \infty. \tag{3.2}$$

(2) The random variable ξ is $\mathscr{F}_T$-measurable and square-integrable. The lower reflecting boundary L is continuous, progressively measurable, and satisfies

$$\mathbb{E}\left[\sup_{[0,T]} L^+(t)^2\right] < \infty. \tag{3.3}$$

Also, $L(T) \le \xi$, a.e. on Ω.

Under Assumption 3.1, this section proves existence and uniqueness of solution (Y,Z,K) to the following BSDE

$$\begin{cases} Y(t) = \xi + \displaystyle\int_t^T g(s,Y(s),Z(s))ds - \int_t^T Z(s)dB_s + K(T) - K(t); \\ Y(t) \ge L(t),\ 0 \le t \le T,\ \displaystyle\int_0^T (Y(t)-L(t))dK(t) = 0, \end{cases} \tag{3.4}$$

in the spaces

$$
\begin{aligned}
&Y(\cdot) = (Y_1(\cdot), \cdots, Y_m(\cdot))' \in \mathbb{M}^2(m; 0, T)\\
=&\left\{ m\text{-dimensional predictable process } \phi(\cdot) \text{ s.t. } \mathbb{E}\left[\sup_{[0,T]} \phi_t^2\right] \leq \infty \right\};\\
&Z(\cdot) = (Z_1(\cdot), \cdots, Z_m(\cdot))' \in \mathbb{L}^2(m \times d; 0, T)\\
=&\left\{ m \times d\text{-dimensional predictable process } \phi(\cdot) \text{ s.t. } \mathbb{E}\left[\int_0^T \phi_t^2 dt\right] \leq \infty \right\};\\
&K(\cdot) = (K_1(\cdot), \cdots, K_m(\cdot))'\text{: continuous, increasing process in } \mathbb{M}^2(m; 0, T),
\end{aligned}
\tag{3.5}
$$

where the positive integer m is the dimension of the equation. The backward equation and the reflecting condition in (3.4) should be interpreted component-wise; for every $i = 1, \cdots, m$, we have

$$
\begin{cases}
Y_i(t) = \xi_i + \int_t^T g_i(s, Y(s), Z(s))ds - \int_t^T Z_i(s)dB_s + K_i(T) - K_i(t);\\
Y_i(t) \geq L_i(t),\ 0 \leq t \leq T,\ \int_0^T (Y_i(t) - L_i(t))dK_i(t) = 0.
\end{cases}
\tag{3.6}
$$

The value process $Y_i(\cdot)$ is driven by the Brownian noise $B(\cdot)$, whose intensity is modulated by a "control" $Z_i(\cdot)$. The driver g_i leads the value $Y_i(\cdot)$ towards the "final destination" ξ_i. Whenever the ith component $Y_i(\cdot)$ drops to the lower reflecting boundary $L_i(\cdot)$, it receives a force $K_i(\cdot)$ that kicks it upwards. When $Y_i(\cdot)$ stays above level $L_i(\cdot)$, the force $K_i(\cdot)$ does not apply. The process $K_i(\cdot)$ stands for the minimum cumulative exogenous energy required to keep $Y_i(\cdot)$ above level $L_i(\cdot)$. The m equations compose a system of m "vehicles" whose "drivers" track each other. For notational simplicity, the vector form (3.4) is used as a shorthand.

Lemma 3.1 *For any processes $(Y^0(\cdot), Z^0(\cdot)) \in \mathbb{L}^2(m; 0, T) \times \mathbb{L}^2(m \times d; 0, T)$, there exist unique $(Y^1(\cdot), Z^1(\cdot)) \in \mathbb{M}^2(m; 0, T) \times \mathbb{L}^2(m \times d; 0, T)$, and $K^1(\cdot) \in \mathbb{M}^2(m; 0, T)$, such that*

$$
\begin{cases}
dY^1(t) = -g(t, Y^0(t), Z^0(t))dt + Z^1(t)dB_t - dK^1(t),\ 0 \leq t \leq T;\\
Y^1(T) = \xi;\\
Y^1(t) \geq L(t),\ 0 \leq t \leq T,\ \int_0^T (Y^1(t) - L(t))dK^1(t) = 0.
\end{cases}
\tag{3.7}
$$

Proof. For any $i = 1, \cdots, m$, in the ith dimension, by Corollary 3.7 of El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) [12], there exists a unique solution $(Y_i^1(\cdot), Z_i^1(\cdot)) \in \mathbb{M}^2(1; 0, T) \times \mathbb{L}^2(d; 0, T)$, and a continuous, increasing process $K_i^1(\cdot) \in \mathbb{M}^2(1; 0, T)$, to the one-dimensional reflected BSDE

$$
\begin{cases}
dY_i^1(t) = -g_i(t, Y^0(t), Z^0(t))dt + Z_i^1(t)dB_t - dK_i^1(t),\ 0 \leq t \leq T;\\
Y_i^1(T) = \xi_i;\\
Y_i^1(t) \geq L_i(t),\ 0 \leq t \leq T,\ \int_0^T (Y_i^1(t) - L_i(t))dK_i^1(t) = 0.
\end{cases}
\tag{3.8}
$$

The processes $Y^1(\cdot) := (Y_1^1(\cdot), \cdots, Y_m^1(\cdot))'$, $Z^1(\cdot) := (Z_1^1(\cdot), \cdots, Z_m^1(\cdot))'$, and $K^1(\cdot) := (K_1^1(\cdot), \cdots, K_m^1(\cdot))'$ form the desired triple. $\square$

To prove existence and uniqueness of the solution to the multi-dimensional BSDE (3.4) with reflecting barrier, it suffices to show that the mapping

$$\begin{aligned}\Lambda :&\mathbb{L}^2(m;0,T) \times \mathbb{L}^2(m \times d;0,T) \to \mathbb{L}^2(m;0,T) \times \mathbb{L}^2(m \times d;0,T);\\ &(Y^0, Z^0) \mapsto (Y^1, Z^1)\end{aligned} \tag{3.9}$$

is a contraction.

Theorem 3.1 *The mapping Λ is a contraction from $\mathbb{L}^2(m;0,T) \times \mathbb{L}^2(m \times d;0,T)$ to $\mathbb{L}^2(m;0,T) \times \mathbb{L}^2(m \times d;0,T)$.*

Proof. For a progressively measurable process $\phi(\cdot)$, the norm $||\phi||_2 := \sqrt{\mathbb{E}\Big[\int_0^T \phi_t^2 dt\Big]}$ is equivalent to the norm $||\phi||_{2,\beta} := \sqrt{\mathbb{E}\Big[\int_0^T e^{\beta t}\phi_t^2 dt\Big]}$. We prove the contraction statement under the norm $||\cdot||_{2,\beta}$. Suppose $(Y^0(\cdot), Z^0(\cdot))$ and $(\bar{Y}^0(\cdot), \bar{Z}^0(\cdot))$ are both in $\mathbb{M}^2(m;0,T) \times \mathbb{L}^2(m \times d;0,T)$. Denote $(Y^1(\cdot), Z^1(\cdot)) = \Lambda(Y^0(\cdot), Z^0(\cdot))$ and $(\bar{Y}^1(\cdot), \bar{Z}^1(\cdot)) = \Lambda(\bar{Y}^0(\cdot), \bar{Z}^0(\cdot))$. Applying Itô's rule to $e^{\beta t}(Y^1(t) - \bar{Y}^1(t))^2$, integrating the derivative from t to T, using the uniform Lipschitz condition, Assumption 3.1 (1) of g, and applying some elementary inequalities, we get that after taking expectation,

$$||Y^1 - \bar{Y}^1||_{2,\beta}^2 + ||Z^1 - \bar{Z}^1||_{2,\beta}^2 \leq \frac{1}{2}||Y^0 - \bar{Y}^0||_{2,\beta}^2 + \frac{1}{2}||Z^0 - \bar{Z}^0||_{2,\beta}^2. \tag{3.10}$$

$\square$

Proposition 3.1 *The BSDE (3.4) with reflecting barrier has a unique solution in $\mathbb{M}^2(m;0,T) \times \mathbb{L}^2(m \times d;0,T)$.*

Proof. The solution is the unique fixed-point, say $(Y(\cdot), Z(\cdot))$, of the contraction Λ. Since $(Y(\cdot), Z(\cdot)) \in \mathbb{L}^2(m;0,T) \times \mathbb{L}^2(m \times d;0,T)$, $(Y(\cdot), Z(\cdot)) = \Lambda(Y(\cdot), Z(\cdot))$ is also in $\mathbb{M}^2(m;0,T) \times \mathbb{L}^2(m \times d;0,T)$ by Lemma 3.1. $\square$

Theorem 3.2 *(Comparison Theorem, El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) [12])*
Suppose (Y, Z, K) solves (3.4) with parameter set (ξ, g, L), and $(\bar{Y}, \bar{Z}, \bar{K})$ solves (3.4) with parameter set $(\bar{\xi}, \bar{g}, \bar{L})$. Let dimension of the equations be $m = 1$. Under Assumption 3.1, except that the uniform Lipschitz condition only needed for either g or $\bar{g}$, if
(1) $\xi \leq \bar{\xi}$, a.e.;
(2) $g(t, y, z) \leq \bar{g}(t, y, z)$, a.e. $(t, \omega) \in [0, T] \times \Omega$, $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$; and
(3) $L(t) \leq \bar{L}(t)$, a.e. $(t, \omega) \in [0, T] \times \Omega$,
then

$$Y(t) \leq \bar{Y}(t), \ a.e. \ (t, \omega) \in [0, T] \times \Omega. \tag{3.11}$$

Theorem 3.3 *(Continuous Dependence Property)*
Under Assumption 3.1, suppose that (Y, Z, K) solves RBSDE (3.4), and that $(\bar{Y}, \bar{Z}, \bar{K})$ solves

$$\begin{cases} \bar{Y}(t) = \bar{\xi} + \int_t^T g(s, \bar{Y}(s), \bar{Z}(s))ds - \int_t^T \bar{Z}(s)dB_s + \bar{K}(T) - \bar{K}(t); \\ \bar{Y}(t) \geq L(t),\ 0 \leq t \leq T,\ \int_0^T (\bar{Y}(t) - L(t))d\bar{K}(t) = 0, \end{cases} \tag{3.12}$$

then there exists a constant number C, such that for all $0 \leq t \leq T$,

$$\begin{aligned} &\mathbb{E}[(Y(t) - \bar{Y}(t))^2] + \mathbb{E}\left[\int_0^T (Y(s) - \bar{Y}(s))^2 ds\right] \\ &+ \mathbb{E}\left[\int_0^T (Z(s) - \bar{Z}(s))^2 ds\right] + \mathbb{E}[(K(t) - \bar{K}(t))^2] \\ \leq &C\mathbb{E}[(\xi - \bar{\xi})^2]. \end{aligned} \tag{3.13}$$

Proof. Repeating the methods in the proof of Theorem 3.1, we can show that both

$$\mathbb{E}[(Y(t) - \bar{Y}(t))^2] \leq e^{\beta T}\mathbb{E}[(\xi - \bar{\xi})^2], \text{ for all } 0 \leq t \leq T, \tag{3.14}$$

and

$$\mathbb{E}\left[\int_0^T (Y(s) - \bar{Y}(s))^2 ds\right] + \mathbb{E}\left[\int_0^T (Z(s) - \bar{Z}(s))^2 ds\right] \leq 2e^{\beta T}\mathbb{E}\left[(\xi - \bar{\xi})^2\right] \tag{3.15}$$

hold true.
From the expressions

$$K(t) = Y(0) - Y(t) - \int_0^t g(s, Y(s), Z(s))ds + \int_0^t Z(s)dB_s, \tag{3.16}$$

and

$$\bar{K}(t) = \bar{Y}(0) - \bar{Y}(t) - \int_0^t g(s, \bar{Y}(s), \bar{Z}(s))ds + \int_0^t \bar{Z}(s)dB_s. \tag{3.17}$$

By the Lipschitz condition, Assumption 3.1 (1), and Itô's isometry, for all $0 \leq t \leq T$, we derive the following estimation for the $\mathbb{L}^2$-norm of $(K(t) - \bar{K}(t))$,

$$\begin{aligned} &\mathbb{E}\left[(K(t) - \bar{K}(t))^2\right] \\ \leq &C_1\Bigg(\mathbb{E}\left[(Y(0) - \bar{Y}(0))^2\right] + \mathbb{E}\left[(Y(t) - \bar{Y}(t))^2\right] + 2Tb^2\mathbb{E}\left[\int_0^T (Y(t) - \bar{Y}(t))^2 dt\right] \\ &+ (2Tb^2 + 1)\mathbb{E}\left[\int_0^T (Z(t) - \bar{Z}(t))^2 dt\right]\Bigg) \\ \leq &4C_1(Tb^2 + 1)e^{\beta T}\mathbb{E}\left[(\xi - \bar{\xi})^2\right], \end{aligned} \tag{3.18}$$

where the last inequality follows from (3.14) and (3.15). □

4 Markovian System with Linear Growth Rate

This section shows the existence of solution to the multi-dimensional BSDE with reflecting barrier within a Markovian framework. The growth rate of the forward equation is assumed polynomial in the state process X, and linear in both the value process Y and the volatility process Z. The comparison theorem in dimension one, and the continuous dependence property of the value process and the volatility process on the terminal condition, are also provided.

The Markovian system of forward-backward SDEs in question is the following pair of equations.

$$\begin{cases} X^{t,x}(s) = x, \ 0 \leq s \leq t; \\ dX^{t,x}(s) = f(s, X^{t,x}(s))ds + \sigma(s, X^{t,x}(s))dB_s, \ t < s \leq T. \end{cases} \tag{4.1}$$

$$\begin{cases} Y^{t,x}(s) = \xi(X^{t,x}(T)) + \int_s^T g(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r))dr - \int_s^T Z^{t,x}(r)dB_r \\ \qquad + K^{t,x}(T) - K^{t,x}(s); \\ Y^{t,x}(s) \geq L(s, X^{t,x}(s)), \ t \leq s \leq T, \ \int_t^T (Y^{t,x}(s) - L(s, X^{t,x}(s)))dK^{t,x}(s) = 0. \end{cases} \tag{4.2}$$

For any $x \in \mathbb{R}^l$, the SDE (4.1) has a unique strong solution, under Assumption 4.1 (1) below (cf. page 287, Karatzas and Shreve (1988) [28]). A solution to the forward-backward system (4.1) and (4.2) is a triple of processes $(Y^{t,x}, Z^{t,x}, K^{t,x})$ satisfying (4.2), where $Y^{t,x} \in \mathbb{M}^2(m; 0, T)$, $Z^{t,x} \in \mathbb{L}^2(m \times d; 0, T)$, and $K^{t,x}$ is a continuous, increasing process in $\mathbb{M}^2(m; 0, T)$. The superscript (t, x) on X, Y, Z, and K indicates the state x of the underlying process X at time t. It will be omitted for notational simplicity.

Assumption 4.1 *(1) In (4.1), the drift $f : [0, T] \times \mathbb{R}^l \to \mathbb{R}^l$, and volatility $\sigma : [0, T] \times \mathbb{R}^l \to \mathbb{R}^{l \times d}$, are deterministic, measurable mappings, locally Lipschitz in x uniformly over all $t \in [0, T]$. And for all $(t, x) \in [0, T] \times \mathbb{R}^l$, $|f(t, x)|^2 + |\sigma(t, x)|^2 \leq C(1 + |x|^2)$, for some constant C.*
(2) In (4.2), the driver g is a deterministic measurable mapping $g : [0, T] \times \mathbb{R}^l \times \mathbb{R}^m \times^{m \times d} \to \mathbb{R}^m$, $(t, x, y, z) \mapsto g(t, x, y, z)$. And for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, $|g(t, x, y, z)| \leq b(1 + |x|^p + |y| + |z|)$, for some positive constant b.
(3) For every fixed $(t, x) \in [0, T] \times \mathbb{R}^l$, the mapping $g(t, x, \cdot, \cdot)$ is continuous.
(4) The terminal value $\xi : \mathbb{R}^l \to \mathbb{R}^m$, $x \mapsto \xi(x)$, is a deterministic measurable mapping. The lower reflecting boundary $L : [0, T] \times \mathbb{R}^l \to \mathbb{R}^m$, $(s, x) \mapsto L(s, x)$ is deterministic measurable mapping continuous in (s, x). They satisfy $\mathbb{E}[\xi(X(T))^2] < \infty$, $\mathbb{E}\left[\sup_{[0,T]} L^+(s, X(s))^2\right] < \infty$, and $L(T, X(T)) \leq \xi(X(T))$, a.e. on Ω.

Theorem 4.1 *Suppose that Assumption 4.1 holds, except the growth rate condition on g. If the driver $g(s, x, y, z)$ in the reflected BSDE (4.2) is Lipschitz in y and z, uniform for all $s \in [0, T]$ and all $x \in \mathbb{R}^l$, then there exist measurable deterministic functions $\alpha : [0, T] \times \mathbb{R}^l \to \mathbb{R}^m$, and $\beta : [0, T] \times \mathbb{R}^l \to \mathbb{R}^{m \times d}$, such that for any $0 \leq t \leq s \leq T$, $Y^{t,x}(s) = \alpha(s, X^{t,x}(s))$, and $Z^{t,x}(s) = \beta(s, X^{t,x}(s))$. The solutions to the BSDE are functions of the state process X.*

Proof. First, the one-dimensional case $m = 1$. There exist measurable, deterministic functions $a^n : [0,T] \times \mathbb{R}^l \to \mathbb{R}$, $b^n : [0,T] \times \mathbb{R}^l \to \mathbb{R}^d$, such that for any $0 \le t \le s \le T$, the solution $(Y^{(t,x),n}, Z^{(t,x),n})$ to the penalized equation

$$\begin{aligned} Y^{(t,x),n}(s) =& \xi(X^{t,x}(T)) + \int_s^T g(r, X^{t,x}(r), Y^{(t,x),n}(r), Z^{(t,x),n}(r))dr - \int_s^T Z^{(t,x),n}(r)dB_r \\ &+ n\int_s^T (Y^{(t,x),n}(r) - L(r, X^{t,x}(r)))^- dr \end{aligned} \tag{4.3}$$

can be expressed as $Y^{(t,x),n}(s) = a^n(s, X^{t,x}(s))$, and $Z^{(t,x),n}(s) = b^n(s, X^{t,x}(s))$; in particular, $Y^{(t,x),n}(t) = a^n(t,x)$. This is the Markovian property of solutions to one one-dimensional forward-backward SDEs with Lipschitz driver, stated as Theorem 4.1 in El Karoui, Peng and Quenez (1997) [13]. Their proof uses the Picard iteration and the Markov property of the iterated sequence of solutions, the latter being an interpretation of Theorem 6.27 on page 206 of Çinlar, Jacod, Protter and Sharpe (1980) [7]. Analyzed in section 6, El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) [12], its solution $(Y^{(t,x),n}, Z^{(t,x),n})$ converges to some limit $(Y^{t,x}, Z^{t,x})$ in $\mathbb{M}^2(m;t,T) \times \mathbb{L}^2(m \times d; t, T)$. The penalization term $n\int_0^s (Y^{(t,x),n}(r) - L(r, X^{t,x}(r)))^- dr$ also has an $\mathbb{M}^2(m;0,T)$-limit $K^{t,x}(s)$. The triple $(Y^{t,x}, Z^{t,x}, K^{t,x})$ solves the system (4.1) and (4.2). But the convergences are also almost everywhere on $\Omega \times [t,T]$, so

$$Y^{t,x}(s) = \lim_{n\to\infty} Y^{(t,x),n}(s) = \limsup_{n\to\infty}(a^n(s, X^{t,x}(s))) = \limsup_{n\to\infty}(a^n)(s, X^{t,x}(s)) =: a(s, X^{t,x}(s)), \tag{4.4}$$

and

$$Z^{t,x}(s) = \lim_{n\to\infty} Z^{(t,x),n}(s) = \limsup_{n\to\infty}(b^n(s, X^{t,x}(s))) = \limsup_{n\to\infty}(b^n)(s, X^{t,x}(s)) =: b(s, X^{t,x}(s)). \tag{4.5}$$

Back to a general dimension m. By Theorem 3.1 and Proposition 3.1, the sequence $(Y^{n+1}, Z^{n+1}) = \Lambda(Y^n, Z^n)$, $n = 0, 1, 2, \cdots$, iterated via the mapping Λ as in (3.1), converges to (Y, Z) a.e. on $\Omega \times [t,T]$ and in $\mathbb{M}^2(m;0,T) \times \mathbb{L}^2(m \times d; 0,T)$. If one can prove $Y^1(s)$ and $Z^1(s)$ are functions of $(s, X(s))$, so is every $(Y^n(s), Z^n(s))$ by induction. Then the theorem holds, because (Y, Z) is the pointwise limit of $\{(Y^n(s), Z^n(s))\}_{n\in\mathbb{N}}$. The claim is indeed true. Starting with $Y^{(t,x),0}(s) = \alpha^0(s, X(s))$, and $Z^{(t,x),0}(s) = \beta^0(s, X(s))$, for any measurable, deterministic functions $\alpha^0 : [0,T] \times \mathbb{R}^l \to \mathbb{R}^m$, and $\beta^0 : [0,T] \times \mathbb{R}^l \to \mathbb{R}^{m\times d}$ satisfying $\alpha^0(\cdot, X^{t,x}(\cdot)) \in \mathbb{M}^2(m;0,T)$, and $\beta^0(\cdot, X^{t,x}(\cdot)) \in \mathbb{L}^2(m \times d; 0, T)$. In an arbitrary ith dimension, $1 \le i \le m$,

$$\begin{cases} Y_i^1(s) =\xi_i(X^{t,x}(T)) + \displaystyle\int_s^T g_i(r, X^{t,x}(r), \alpha^0(r, X(r)), \beta^0(r, X(r)))dr \\ \qquad\qquad - \displaystyle\int_s^T Z_i^1(r)dB_r + K_i^1(T) - K_i^1(s); \\ Y_i^1(s) \ge L_i(s, X^{t,x}(s)),\ t \le s \le T,\ \displaystyle\int_t^T (Y_i^1(s) - L_i(s, X^{t,x}(s)))dK_i^1(s) = 0. \end{cases} \tag{4.6}$$

From the one-dimensional result, there exist measurable, deterministic functions $\alpha_i^1 : [0,T] \times \mathbb{R}^l \to \mathbb{R}$, and $\beta_i^1 : [0,T] \times \mathbb{R}^l \to \mathbb{R}^d$, such that $Y_i^{(t,x),1}(s) = \alpha_i^1(s, X^{t,x}(s))$, and $Z_i^{(t,x),1}(s) = \beta_i^1(s, X^{t,x}(s))$, for all $0 \le t \le s \le T$. Let $\alpha^1 = (\alpha_1^1, \cdots, \alpha_m^1)'$, and $\beta^1 = (\beta_1^1, \cdots, \beta_m^1)'$, then $Y^{(t,x),1}(s) = \alpha^1(s, X^{t,x}(s))$, and $Z^{(t,x),1}(s) = \beta^1(s, X^{t,x}(s))$, for all $0 \le t \le s \le T$. $\square$

Remark 4.1 *To prove the above theorem, besides using the notion of additive martingales as in Çinlar et al (1980) [7], the two deterministic functions can also be obtained by solving a multi-dimensional variational inequality following the four-step-scheme proposed by Ma, Protter and Yong (1994) [33].*

The rest of this section will be devoted to proving the existence of solutions to the reflected forward-backward system (4.1) and (4.2) under the Assumption 4.1. We shall construct a specific sequence of Lipschitz drivers $\{g^n\}_{n\in\mathbb{N}}$ to approximate the linear-growth driver g. The corresponding sequence of solutions will turn out to converge to the system (4.1) and (4.2). We then approximate the continuous linear growth driver g by a sequence of Lipschitz functions g^n.

Let $\bar{\psi}$ be an infinitely differentiable mapping from $\mathbb{R}^m \times \mathbb{R}^{m\times d}$ to $\mathbb{R}$, such that

$$\bar{\psi}(y,z) = \begin{cases} 1, & |y|^2 + |z|^2 \le 1; \\ 0, & |y|^2 + |z|^2 \ge 4, \end{cases} \tag{4.7}$$

and ψ a rescaling of $\bar{\psi}$ by a multiplicative constant such that

$$\int_{\mathbb{R}^m \times \mathbb{R}^{m\times d}} \psi(y,z) dy dz = 1. \tag{4.8}$$

The function ψ is a kernel conventionally used to smooth out non-differentiability, for example, by Karatzas and Ocone (1992) [27], or to approximate functions of higher growth rate, for example, by Hamadène, Lepeltier and Peng (1997) [19].

The approximating sequence g^n is defined as

$$g^n(t,x,y,z) = n^2 \psi\left(\frac{y}{n}, \frac{z}{n}\right) \int_{\mathbb{R}^m \times \mathbb{R}^{m\times d}} g(t,x,y_1,z_1) \bar{\psi}(n(y-y_1), n(z-z_1)) dy_1 dz_1. \tag{4.9}$$

According to Hamadène, Lepeltier and Peng (1997) [19], the sequence of functions g^n has the properties:
(a) g^n is Lipschitz with respect to (y,z), uniformly over all $(t,x) \in [0,T] \times \mathbb{R}^l$;
(b) $|g^n(t,x,y,z)| \le b(1 + |x|^p + |y| + |z|)$, for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, for some positive constant b;
(c) $|g^n(t,x,y,z)| \le b_n(1+|x|^p)$, for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, for some positive constant b_n;
(d) for any $(t,x) \in [0,T] \times \mathbb{R}$, and for any compact set $S \subset \mathbb{R}^m \times \mathbb{R}^{m\times d}$,

$$\sup_{(y,z)\in S} |g^n(t,x,y,z) - g(t,x,y,z)| \to 0, \text{ as } n \to 0. \tag{4.10}$$

Proposition 4.1 *The BSDE with reflecting barrier*

$$\begin{cases} Y^n(s) = \xi(X(T)) + \int_s^T g^n(r, X(r), Y^n(r), Z^n(r))dr - \int_s^T Z^n(r)dB_r + K^n(T) - K^n(s); \\ Y^n(s) \geq L(s, X(s)),\ t \leq s \leq T,\ \int_t^T (Y^n(s) - L(s, X(s)))dK^n(s) = 0 \end{cases} \tag{4.11}$$

has a unique solution (Y^n, Z^n, K^n)*. Furthermore, there exist measurable, deterministic functions* α^n *and* β^n*, such that* $Y^n(s) = \alpha^n(s, X(s))$*, and* $Z^n(s) = \beta^n(s, X(s))$*, for all* $0 \leq s \leq T$*.*

Proof. This is a direct consequence of the uniform Lipschitz property of g^n, Proposition 3.1 and Theorem 4.1. □

Lemma 4.1 *Suppose* (Y, Z, K) *solves the BSDE (4.2) with reflecting barrier. Under the conditions (2) and (4) of Assumption 4.1, there exists a positive constant* C *such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq T} Y(s)^2 + \int_t^T Z(r)^2 ds + K(T)^2\right] \leq C(1 + |x|^{2(p\vee 1)}). \tag{4.12}$$

The constant C *does not depend on* t*, but depends on* m*,* b*,* T*,* $\mathbb{E}[\xi(X(T))^2]$ *and* $\mathbb{E}\left[\sup_{[0,T]} L^+(t, X(t))^2\right]$.

Proof. First prove that, for some constant C', we have

$$\mathbb{E}\left[Y(s)^2 + \int_t^T Z(r)^2 ds + K(T)^2\right] \leq C'(1 + |x|^{2(p\vee 1)}), \text{ for all } 0 \leq s \leq T. \tag{4.13}$$

Applying Itô's rule to $Y(\cdot)^2$, and integrating from s to T, we get

$$\begin{aligned} &Y(s)^2 + \int_s^T Z(r)^2 dr \\ =&\xi(X(T))^2 + 2\int_s^T Y(r)g(r, X(r), Y(r), Z(r))dr - 2\int_s^T Y(r)Z(r)dB(r) + 2\int_s^T L(r, X(r))dK(r). \end{aligned} \tag{4.14}$$

Taking expectations of (4.14), and using Assumption 4.1 (2), we obtain

$$\begin{aligned} &\mathbb{E}\left[Y(s)^2 + \int_s^T Z(r)^2 dr\right] \\ \leq&\mathbb{E}[\xi(X(T))^2] + 2b\mathbb{E}\left[\int_s^T |Y(r)|(1 + |X(r)|^p + |Y(r)| + |Z(r)|)dr\right] \\ &+ 2\mathbb{E}\left[\int_s^T L(r, X(r))dK(r)\right] \\ \leq&\mathbb{E}[\xi(X(T))^2] + 2\mathbb{E}\left[\int_s^T (1 + |X(r)|^{2p})dr\right] + C_1(b)\mathbb{E}\left[\int_s^T |Y(r)|^2 dr\right] \\ &+ \frac{1}{4}\mathbb{E}\left[\int_s^T |Z(r)|^2 dr\right] + 2\mathbb{E}\left[\int_s^T L(r, X(r))dK(r)\right]. \end{aligned} \tag{4.15}$$

For any $\alpha > 0$,

$$2\int_t^T L(s,X(s))dK(s) \leq 2\left(\sup_{[0,T]} L(s,X(s))\right) K(T) \leq \frac{1}{\alpha}K(T)^2 + \alpha \sup_{[0,T]} L^+(s,X(s))^2. \tag{4.16}$$

Combine (4.15) and (4.16), and apply Gronwall's Lemma to $Y(\cdot)$,

$$\begin{aligned}
&\mathbb{E}\left[Y(s)^2 + \frac{3}{4}\int_s^T Z(r)^2 dr\right] \\
\leq & C_2(b,T)\left(1 + \mathbb{E}[\xi(X(T))^2] + \mathbb{E}\left[\int_s^T |X(r)|^{2p} dr\right] + \frac{1}{\alpha}K(T)^2 + \alpha \sup_{[0,T]} L^+(s,X(s))^2\right).
\end{aligned} \tag{4.17}$$

If rewriting (4.2) from t to T, $K(\cdot)$ can be expressed in terms of $Y(\cdot)$ and $Z(\cdot)$ by

$$K(T) = Y(t) - \xi(X(T)) - \int_t^T g(s,X(s),Y(s),Z(s))ds + \int_t^T Z(s)dB_s, \tag{4.18}$$

and hence because of the linear growth Assumption 4.1 (2), we have

$$\begin{aligned}
&\mathbb{E}[K(T)^2] = C_3\mathbb{E}\left[Y(t)^2 + \xi(X(T))^2 + \int_t^T g(s,X(s),Y(s),Z(s))^2 ds + \int_t^T Z(s)^2 ds\right] \\
\leq & C_4(b)\Bigg(\mathbb{E}\left[Y(t)^2 + \xi(X(T))^2 + 1 + \int_t^T |X(s)|^{2p} ds\right] + \mathbb{E}\left[\int_t^T |Y(s)|^2 ds\right] \\
&+ \mathbb{E}\left[\int_t^T |Z(s)|^2 ds\right]\Bigg).
\end{aligned} \tag{4.19}$$

On the strength of the bounds on $\mathbb{E}[|Y(s)|^2]$ and $\mathbb{E}\left[\int_t^T |Z(s)|^2 ds\right]$ obtained in (4.17), we deduce from (4.19):

$$\begin{aligned}
\mathbb{E}[K(T)^2] \leq C_5(b,t,T)\Bigg(&\mathbb{E}\left[\xi(X(T))^2 + 1 + \int_t^T |X(s)|^{2p} ds\right] \\
&+ \frac{1}{\alpha}\mathbb{E}[K(T)^2] + \alpha\mathbb{E}\left[\sup_{[0,T]} L^+(s,X(s))^2\right]\Bigg).
\end{aligned} \tag{4.20}$$

Let $\alpha = 4C_5(b,t,T)$, and collect $\mathbb{E}[K(T)^2]$ terms on both sides of (4.20),

$$\mathbb{E}[K(T)^2] \leq C_6(b,t,T)\mathbb{E}\left[\xi(X(T))^2 + 1 + \int_t^T |X(s)|^{2p} ds + \sup_{[0,T]} L^+(s,X(s))^2\right]. \tag{4.21}$$

Finally, (4.17) and (4.21) altogether gives

$$\begin{aligned}&\mathbb{E}\left[Y(s)^2+\int_s^T Z(r)^2ds+K(T)^2\right]\\ \leq&C_7(b,t,T)\left(1+\mathbb{E}[\xi(X(T))^2]+\mathbb{E}\left[\int_t^T|X(r)|^{2p}dr\right]+\mathbb{E}\left[\sup_{[0,T]}L^+(s,X(s))^2\right]\right).\end{aligned}\tag{4.22}$$

From page 306 of Karatzas and Shreve (1988) [28], for $p\geq 1$,

$$\mathbb{E}\left[\sup_{[0,T]}|X^{t,x}(s)|^{2p}\right]\leq C_8(1+|x|^{2p}).\tag{4.23}$$

Then the constant C' in (4.13) can be chosen as

$$C'=\left(\sup_{0\leq t\leq T}C_7(b,t,T)\right)\max\left\{1+\mathbb{E}[\xi(X(T))^2]+\mathbb{E}\left[\sup_{[0,T]}L^+(s,X(s))^2\right],C_8T\right\}<\infty.\tag{4.24}$$

To bound the $\mathbb{L}^2$ supremum norm of $Y(\cdot)$, taking first supremum over $s\in[0,T]$ then expectation, on both sides of (4.14), using Burkholder-Davis-Gundy inequality, and combining with (4.16),

$$\begin{aligned}&\mathbb{E}\left[\sup_{[0,T]}Y(s)^2+\int_t^T Z(r)^2dr\right]\\ \leq&\mathbb{E}[\xi(X(T))^2]+2b\mathbb{E}\left[\sup_{[0,T]}\int_s^T|Y(r)|(1+|X(r)|^p+|Y(r)|+|Z(r)|)dr\right]\\ &+C_9(m)\mathbb{E}\left[\sqrt{\int_t^T|Y(r)|^2\cdot|Z(r)|^2dr}\right]+2\mathbb{E}\left[\int_s^T L(r,X(r))dK(r)\right]\\ \leq&\mathbb{E}[\xi(X(T))^2]+b\mathbb{E}\left[\int_t^T|Y(r)|^2dr\right]+b\mathbb{E}\left[\int_t^T(1+|X(r)|^p+|Y(r)|+|Z(r)|)^2dr\right]\\ &+C_9(m)\mathbb{E}\left[\sup_{[0,T]}|Y(s)|\sqrt{\int_t^T|Z(r)|^2dr}\right]+\mathbb{E}[K(T)^2]+\mathbb{E}\left[\sup_{[0,T]}L^+(s,X(s))^2\right]\\ \leq&\mathbb{E}[\xi(X(T))^2]+C_{10}(b)\mathbb{E}\left[\int_t^T(1+|X(r)|^{2p}+|Y(r)|^2+|Z(r)|^2)dr\right]\\ &+\frac{1}{2}\mathbb{E}\left[\sup_{[0,T]}|Y(s)|^2\right]+2C_9(m)^2\mathbb{E}\left[\int_t^T|Z(r)|^2dr\right]+\mathbb{E}[K(T)^2]+\mathbb{E}\left[\sup_{[0,T]}L^+(s,X(s))^2\right].\end{aligned}\tag{4.25}$$

Equation (4.25) implies that

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\sup_{[0,T]} Y(s)^2\right] \\
\leq &\mathbb{E}[\xi(X(T))^2] + C_{10}(b)\mathbb{E}\left[\int_t^T (1+|X(r)|^{2p}+|Y(r)|^2+|Z(r)|^2)dr\right] \\
&+ 2C_9(m)^2\mathbb{E}\left[\int_t^T |Z(r)|^2 dr\right] + \mathbb{E}[K(T)^2] + \mathbb{E}\left[\sup_{[0,T]} L^+(s,X(s))^2\right].
\end{aligned}
\tag{4.26}
$$

Inequalities (4.13), (4.23) and (4.26) conclude the lemma. □

Proposition 4.2 *There exists a positive constant C, such that for $0 \leq t \leq T$, $n = 1, 2, \cdots$,*

$$
\alpha^n(t,x) = Y^{(t,x),n}(t) = \mathbb{E}[Y^{(t,x),n}(t)|\mathscr{F}_t] \leq C(1+|x|^{p\vee 1}). \tag{4.27}
$$

Proposition 4.3 *The sequence $\{g^n(\cdot, X(\cdot), Y^n(\cdot), Z^n(\cdot))\}_{n\in\mathbb{N}}$ is uniformly bounded in the $\mathbb{L}^2(m;t,T)$-norm, and the sequence $\{K^n(\cdot)\}_{n\in\mathbb{N}}$ is uniformly bounded in the $\mathbb{M}^2(m;t,T)$-norm, both uniformly over all n. As $n \to \infty$, $g^n(\cdot, X(\cdot), Y^n(\cdot), Z^n(\cdot))$ weakly converges to some limit $G(\cdot)$ in $\mathbb{L}^2(m;t,T)$ along a subsequence, and $K^n(\cdot)$ weakly converges to some limit $K(\cdot)$ in $\mathbb{M}^2(m;t,T)$ along a subsequence, for every $s \in [t,T]$.*

Proof. It suffices to show the uniform boundedness of $\{g^n(\cdot, X(\cdot), Y^n(\cdot), Z^n(\cdot))\}_{n\in\mathbb{N}}$ in $\mathbb{L}^2(m;t,T)$ and of $\{K^n(T)\}_{n\in\mathbb{N}}$ in $\mathbb{L}^2(m)$, which is a result of the linear growth property (b) and Lemma 4.1. The $\mathbb{L}^2(m)$ uniform boundedness of $\{K^n(T)\}_{n\in\mathbb{N}}$ means that there exists a constant $C < \infty$, such that $\mathbb{E}[|K^n(T)|^2] < C$. Since $K^n(\cdot)$ is required to be an increasing process starting from $K^n(t) = 0$, then for all $t \leq s \leq T$, $\mathbb{E}[|K^n(s)|^2] \leq \mathbb{E}[|K^n(T)|^2] < C$. □

With the help of weak convergence along a subsequence, we proceed to argue that the weak limits are also strong, thus deriving a solution to BSDE (4.2). For notational simplicity, the weakly convergent subsequences are still indexed by n. The passing from weak to strong convergence makes use of the Markovian structure of the system described by Theorem 4.1, which states that the valued process $Y^n(s)$ is a deterministic function of time s and state process $X(s)$ only.

Lemma 4.2 *The approximating sequence of solutions $\{(Y^{(t,x),n}, Z^{(t,x),n})\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{L}^2(m;t,T) \times \mathbb{L}^2(m\times d;t,T)$, thus having a limit $(Y^{t,x}, Z^{t,x})$ in $\mathbb{L}^2(m;t,T) \times \mathbb{L}^2(m\times d;t,T)$ and a.e. on $[t,T]\times\Omega$.*

Proof. For any $t \in [0,T]$, any $x \in \mathbb{R}^l$, and any $n = 1, 2, \cdots$, $Y^{(t,x),n}(t) = \alpha^n(t,x)$ is deterministic. First prove the convergence of $\{\alpha^n(t,x)\}_{n\in\mathbb{N}}$ by showing it is Cauchy. For $n_1, n_2 = 1, 2, \cdots$, from equation (4.11) comes the following inequality,

$$
\begin{aligned}
&|\alpha^{n_1}(t,x) - \alpha^{n_2}(t,x)| = |Y^{n_1}(t) - Y^{n_2}(t)| \\
\leq &\left|\mathbb{E}\left[\int_t^T (g^{n_1}(s,X(s),Y^{n_1}(s),Z^{n_1}(s)) - g^{n_2}(s,X(s),Y^{n_2}(s),Z^{n_2}(s)))ds\right]\right| \\
&+ |\mathbb{E}[K^{n_1}(T) - K^{n_2}(T)]| + |\mathbb{E}[K^{n_1}(t) - K^{n_2}(t)]|.
\end{aligned}
\tag{4.28}
$$

By the weak convergence from Proposition 4.3, all the three summands on the right hand side of the above inequality converge to zero, as n_1 and n_2 both go to infinity. Denote the limit of $\alpha^n(t,x)$ as $\alpha(t,x)$, which is consequently deterministic and measurable, because $\alpha^n(\cdot,\cdot)$ is measurable. Theorem 4.1 states that for any $t \leq s \leq T$, $Y^{(t,x),n}(s) = \alpha^n(s, X^{t,x}(s))$. Because of the pointwise convergence of $\alpha^n(\cdot,\cdot)$, $Y^{(t,x),n}(s)$ converges to some $Y^{(t,x)}(s)$, a.e. $(s,\omega) \in [t,T] \times \Omega$, as $n \to \infty$. Proposition 4.2 states that there exists a positive constant C, such that for $0 \leq t \leq T$, $n = 1, 2, \cdots$,

$$|Y^{(t,x),n}(s)| = |\alpha^n(s, X_s^{t,x})| \leq C(1 + |X_s^{t,x}|^{p\vee 1}), \tag{4.29}$$

the last term of which is square-integrable by (4.23). Then it follows from the dominated convergence theorem that the convergence of $Y^{(t,x),n}(s)$ is also in $\mathbb{L}^2(m; t, T)$.

Apply Itô's rule to $(Y^{(t,x),n_1}(s) - Y^{(t,x),n_2}(s))^2$, and integrate from s to T. The reflection conditions in (4.2) gives

$$\begin{aligned}
&(Y^{n_1}(s) - Y^{n_2}(s))^2 + \int_s^T (Z^{n_1}(r) - Z^{n_2}(r))^2 dr \\
\leq &\int_s^T (Y^{n_1}(r) - Y^{n_2}(r))(g^{n_1}(r, X(r), Y^{n_1}(r), Z^{n_1}(r)) - g^{n_2}(r, X(r), Y^{n_2}(r), Z^{n_2}(r)))dr \\
&+ \int_s^T (Y^{n_1}(r) - Y^{n_2}(r))(Z^{n_1}(r) - Z^{n_2}(r))dB_r.
\end{aligned} \tag{4.30}$$

Taking expectation of (4.30),

$$\begin{aligned}
&\mathbb{E}[(Y^{n_1}(s) - Y^{n_2}(s))^2] + \mathbb{E}\left[\int_s^T (Z^{n_1}(r) - Z^{n_2}(r))^2 dr\right] \\
\leq &\mathbb{E}\left[\int_s^T (Y^{n_1}(r) - Y^{n_2}(r))(g^{n_1}(r, X(r), Y^{n_1}(r), Z^{n_1}(r)) - g^{n_2}(r, X(r), Y^{n_2}(r), Z^{n_2}(r)))dr\right] \\
\leq &\mathbb{E}\left[\int_s^T (Y^{n_1}(r) - Y^{n_2}(r))^2 dr\right]^{\frac{1}{2}} \\
&\cdot \mathbb{E}\left[\int_s^T (g^{n_1}(r, X(r), Y^{n_1}(r), Z^{n_1}(r)) - g^{n_2}(r, X(r), Y^{n_2}(r), Z^{n_2}(r)))^2 dr\right]^{\frac{1}{2}}.
\end{aligned} \tag{4.31}$$

In order to prove convergence of $\{Z^n(\cdot)\}_{n\in\mathbb{N}}$, it suffices to prove uniform boundedness of $\mathbb{E}\left[\int_t^T g^n(s, X(s), Y^n(s), Z^n(s))^2 ds\right]$, for all n, which is part of Proposition 4.3. The $\mathbb{L}^2(m \times d; t, T)$-convergence of $\{Z^n\}_{n\in\mathbb{N}}$ implies almost sure convergence along a subsequence, also denoted as $\{Z^n\}_{n\in\mathbb{N}}$ to simplify the notation. □

We have identified a strongly convergent subsequence of $\{(Y^n, Z^n)\}_{n\in\mathbb{N}}$, also denoted as $\{(Y^n, Z^n)\}_{n\in\mathbb{N}}$. Let's remind ourselves that (Y^n, Z^n) solves the system (4.1) and (4.11), so if the weak limit $G(\cdot)$ of $g^n(\cdot, X(\cdot), Y^n(\cdot), Z^n(\cdot))$ is also the strong limit, and if $G(\cdot)$ has the form $g(\cdot, X(\cdot), Y(\cdot), Z(\cdot))$, then the limit (Y, Z, K) indeed solves the forward-backward system (4.1) and (4.2).

Lemma 4.3 *As $n \to \infty$, $g^n(s, X(s), Y^n(s), Z^n(s)) \to g(s, X(s), Y(s), Z(s))$, in $\mathbb{L}^2(m; t, T)$ and a.e. on $[t, T] \times \Omega$.*

Proof. The method is the same as that on page 122 of Hamadène, Lepeltier and Peng (1997) [19]. The proof is briefly repeated here for completeness.

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T |g^n(s, X(s), Y^n(s), Z^n(s)) - g(s, X(s), Y(s), Z(s))| ds\right] \\
\leq &\mathbb{E}\left[\int_t^T |g^n(s, X(s), Y^n(s), Z^n(s)) - g(s, X(s), Y^n(s), Z^n(s))| \mathbb{1}_{\{|Y^n(s)+Z^n(s)| \geq A\}} ds\right] \\
&+ \mathbb{E}\left[\int_t^T |g^n(s, X(s), Y^n(s), Z^n(s)) - g(s, X(s), Y^n(s), Z^n(s))| \mathbb{1}_{\{|Y^n(s)+Z^n(s)| \leq A\}} ds\right] \\
&+ \mathbb{E}\left[\int_t^T |g(s, X(s), Y^n(s), Z^n(s)) - g(s, X(s), Y(s), Z(s))| ds\right].
\end{aligned} \tag{4.32}$$

By linear growth Assumption 4.1 (2) for g and property (b) for g^n, and Lemma 4.1, both

$$|g^n(s, X(s), Y^n(s), Z^n(s)) - g(s, X(s), Y^n(s), Z^n(s))| \tag{4.33}$$

and

$$|g(s, X(s), Y^n(s), Z^n(s)) - g(s, X(s), Y(s), Z(s))| \tag{4.34}$$

are uniformly bounded in $\mathbb{L}^2(m; 0, T)$ for all n. The first term on the right hand side of (4.32) is at most of the order $\frac{1}{A}$, thus vanishing as A goes to infinity. Recalling property (d), for fixed A, the second term vanishes as $n \to \infty$. Because of its uniform boundedness in $\mathbb{L}^2(m; t, T)$, the integrand in the third term is uniformly integrable for all n, so expectation of the integral again goes to 0 as $n \to \infty$.
The a.e. convergent subsequence of $g^n(s, X(s), Y^n(s), Z^n(s))$ is also indexed by n to simplify the notation. □

Proposition 4.4 *The $\mathbb{L}^2(m; t, T)$ convergence and the a.e. convergence of $\{Y^{(t,x),n}(s)\}_{n \in \mathbb{N}}$ to $Y^{(t,x)}(s)$ are uniform over all $s \in [t, T]$.*

Proof. To see uniform convergence of $\{Y^n\}$, applying Itô's rule to $(Y^n(s) - Y(s))^2$, integrating from s to T, taking supremum over $0 \leq s \leq T$ and then expectation, by Burkholder-

Davis-Gundy inequality,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{[0,T]}(Y^n(s)-Y(s))^2\right]+\mathbb{E}\left[\int_t^T(Z^n(r)-Z(r))^2dr\right]\\
\leq&\mathbb{E}\left[\sup_{[0,T]}\int_s^T(Y^n(r)-Y(r))(g^n(r,X(r),Y^n(r),Z^n(r))-g(r,X(r),Y(r),Z(r)))dr\right]\\
&+\mathbb{E}\left[\left(\int_t^T(Y^n(r)-Y(r))^2(Z^n(r)-Z(r))^2dr\right)^{\frac{1}{2}}\right]\\
\leq&\mathbb{E}\Bigg[\left(\int_s^T(g^n(s,X(r),Y^n(r),Z^n(r))-g(r,X(r),Y(r),Z(r)))^2dr\right)^{\frac{1}{2}}\\
&\cdot\sup_{s\in[0,T]}\left(\int_s^T(Y^n(r)-Y(r))^2dr\right)^{\frac{1}{2}}\Bigg]+\mathbb{E}\left[\sup_{s\in[0,T]}\{|Y^n(s)-Y(s)|\}\left(\int_t^T(Z^n(r)-Z(r))^2dr\right)^{\frac{1}{2}}\right]\\
\leq&\left(\mathbb{E}\left[\int_t^T(g^n(r,X(r),Y^n(r),Z^n(r))-g(r,X(r),Y(r),Z(r)))^2dr\right]\right)^{\frac{1}{2}}\\
&\cdot\left(\mathbb{E}\left[\int_t^T(Y^n(r)-Y(r))^2dr\right]\right)^{\frac{1}{2}}+\frac{1}{4}\mathbb{E}\left[\sup_{s\in[0,T]}|Y^n(s)-Y(s)|^2\right]+\mathbb{E}\left[\int_t^T(Z^n(r)-Z(r))^2dr\right].
\end{aligned}
\tag{4.35}
$$

Equation (4.35) implies

$$
\begin{aligned}
&\frac{3}{4}\mathbb{E}\left[\sup_{s\in[0,T]}(Y^n(s)-Y(s))^2\right]\\
\leq&\left(\mathbb{E}\left[\int_t^T(Y^n(r)-Y(r))^2dr\right]\right)^{\frac{1}{2}}\\
&\cdot\left(\mathbb{E}\left[\int_t^T(g^n(r,X(r),Y^n(r),Z^n(r))-g(r,X(r),Y(r),Z(r)))^2dr\right]\right)^{\frac{1}{2}}.
\end{aligned}
\tag{4.36}
$$

By Proposition 4.3, by linear growth properties (b) of g^n and Assumption 4.1 (2) on g, and by Lemma 4.1, the second multiplier on the right hand side of (4.36) is bounded by a constant, uniformly over all n. By Lemma 4.2, the first multiplier on the right hand side of (4.36) converges to zero as $n\to\infty$. Hence

$$
\lim_{n\to\infty}\mathbb{E}\left[\sup_{[0,T]}(Y^n(s)-Y(s))^2\right]=0. \tag{4.37}
$$

□

Proposition 4.5 *The process $K^n(\cdot)$ converges to some limit $K(\cdot)$ in $\mathbb{M}^1(m;t,T)$, uniformly over all $s\in[t,T]$, and a.e. on $[t,T]\times\Omega$.*

Proof. Define

$$\bar{K}(s) := Y(t) - Y(s) - \int_t^s g(r, X(r), Y(r), Z(r))dr + \int_t^s Z(r)dB_r,\ t \leq s \leq T, \tag{4.38}$$

where $Y(\cdot)$, $Z(\cdot)$ and g are the limits of $Y^n(\cdot)$, $Z^n(\cdot)$ and g^n. From (4.11),

$$K^n(s) = Y^n(t) - Y^n(s) - \int_t^s g^n(r, X(r), Y^n(r), Z^n(r))dr + \int_t^s Z^n(r)dB_r. \tag{4.39}$$

Need to show that

$$\mathbb{E}\left[\sup_{s\in[0,T]} \left|K^n(s) - \bar{K}(s)\right|\right] \to 0, \tag{4.40}$$

as $n \to \infty$.

For all $n = 1, 2, \cdots$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,T]} \left|K^n(s) - \bar{K}(s)\right|\right] \\
\leq&\mathbb{E}\left[|Y^n(t) - Y(t)|\right] + \mathbb{E}\left[\sup_{s\in[0,T]} |Y^n(s) - Y(s)|\right] + \mathbb{E}\left[\sup_{s\in[0,T]} \left|\int_t^s (Z^n(r) - Z(r))dB_r\right|\right] \\
&+ \mathbb{E}\left[\int_t^T |g^n(r, X(r), Y^n(r), Z^n(r)) - g(r, X(r), Y(r), Z(r))|\, dr\right].
\end{aligned} \tag{4.41}$$

As $n \to \infty$, the first three summands in (4.41) go to zero, by Lemma 4.2, Proposition 4.4 and Lemma 4.3. From Burkholder-Davis-Gundy inequality, there exists a constant C universal for all n, such that

$$\mathbb{E}\left[\sup_{s\in[0,T]} \left|\int_t^s (Z^n(r) - Z(r))dB_r\right|\right] \leq C\mathbb{E}\left[\left(\int_t^T |Z^n(r) - Z(r)|^2 dr\right)^{\frac{1}{2}}\right], \tag{4.42}$$

the right hand side of which converges to zero as $n \to \infty$, by Lemma 4.2.
The a.e. convergent subsequence is still denoted as $\{K^n(\cdot)\}_{n\in\mathbb{N}}$ to simplify the notation. The strong limit $\bar{K}(\cdot)$ coincides with the weak limit $K(\cdot)$ in Proposition 4.3. □

Proposition 4.6 *The processes $Y(\cdot)$ and $K(\cdot)$ satisfy the reflection conditions $Y(\cdot) \geq L(\cdot, X(\cdot))$ and $\int_t^T (Y(s) - L(s, X(s)))dK(s) = 0$.*

Proof. Since (Y^n, Z^n, K^n) solves (4.11), $Y^n(\cdot)$ and $K^n(\cdot)$ satisfy the reflecting conditions $Y^n(s) \geq L(s, X(s))$, $t \leq s \leq T$, and $\int_t^T (Y^n(s) - L(s, X(s)))dK^n(s) = 0$. Since $Y^n(\cdot)$ converges to $Y(\cdot)$ pointwisely on $[0, T] \times \Omega$, that $Y(\cdot) \geq L(\cdot, X(\cdot))$ holds true. It remains to prove

$$\int_t^T (Y(s) - L(s, X(s)))dK(s) = \int_t^T (Y^n(s) - L(s, X(s)))dK^n(s). \tag{4.43}$$

To wit,

$$\begin{aligned}
&\left|\int_t^T (Y^n(s)-L(s,X(s)))dK^n(s) - \int_t^T (Y(s)-L(s,X(s)))dK(s)\right| \\
\le &\left|\int_t^T (Y^n(s)-Y(s))dK^n(s)\right| + \left|\int_t^T (Y(s)-L(s,X(s)))d(K(s)-K^n(s))\right| \qquad (4.44)\\
\le &\left|\sup_{s\in[0,T]}\{Y^n(s)-Y(s)\}K^n(T)\right| + \left|\int_t^T (Y(s)-L(s,X(s)))d(K(s)-K^n(s))\right|.
\end{aligned}$$

Let n tend to zero. By Proposition 4.4, the first summand in the last line of (4.44) converges to $|0\cdot K(T)|=0$, a.e. on Ω. Proposition 4.5 implies that $K^n(s)$ converges to $K(s)$ in probability, uniformly over all $s\in[t,T]$, so the measure $dK^n(s)$ weakly converges to $dK(s)$ in probability, uniformly over all $s\in[t,T]$. It follows that the second summand in the last line of (4.44) converges to zero, a.e. on Ω. □

We may now conclude the following existence result.

Theorem 4.2 *Under Assumption 4.1, there exists a solution (Y,Z,K) to the BSDE (4.2) with reflecting barrier in the Markovian framework.*

Proof. The solutions $\{(Y^n,Z^n,K^n)\}_{n\in\mathbb{N}}$ to the approximating equations (4.11) have limits (Y,Z,K). The triple (Y,Z,K) is a solution to the Markovian system (4.1) and (4.2). □

Theorem 4.3 *(Comparison Theorem)*
Suppose $(Y^{t,x},Z^{t,x},K^{t,x})$ solves forward-backward system (4.1) and (4.2) with parameter set (ξ,g,L), and $(\bar Y^{t,x},\bar Z^{t,x},\bar K^{t,x})$ solves the forward-backward system (4.1) and (4.2) with parameter set $(\bar\xi,\bar g,\bar L)$. Let dimension of the equations be $m=1$. Under Assumption 4.1 for both sets of parameters, if
(1) $\xi(x)\le\bar\xi(x)$, a.e., $\forall x\in\mathbb{R}^l$;
(2) $g(s,x,y,z)\le\bar g(s,x,y,z)$, for all $t\le s\le T$, and all $(x,y,z)\in\mathbb{R}^l\times\mathbb{R}\times\mathbb{R}^d$; and
(3) $L(s,x)\le\bar L(s,x)$, for all $t\le s\le T$, and all $x\in\mathbb{R}^l$,
then

$$Y^{t,x}(s)\le\bar Y^{t,x}(s),\ \text{for all } t\le s\le T. \qquad (4.45)$$

Proof. Let $\{g^n\}_{n\in\mathbb{N}}$ and $\{\bar g^n\}_{n\in\mathbb{N}}$ be, respectively, the uniform Lipschitz sequences approximating g and $\bar g$ as in (4.9). According to Property (a), both g^n and $\bar g^n$ are Lipschitz in (y,z), for all t and x. We notice that (2) in the conditions of this theorem implies that

$$g^n(s,x,y,z)\le\bar g^n(s,x,y,z), \qquad (4.46)$$

for all $t\le s\le T$, and all $(x,y,z)\in\mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$, via construction (4.9). Let $(Y^{(t,x),n},Z^{(t,x),n},K^{(t,x),n})$ be solution to system (4.1) and (4.2) with parameter set (ξ,g^n,L), and $(\bar Y^{(t,x),n},\bar Z^{(t,x),n},\bar K^{(t,x),n})$ be solution to system (4.1) and (4.2) with parameter set $(\bar\xi,\bar g^n,\bar L)$. By Theorem 3.2,

$$Y^{(t,x),n}(s)\le\bar Y^{(t,x),n}(s),\ t\le s\le T. \qquad (4.47)$$

But as $n \to \infty$, proven earlier in this section,

$$Y^{(t,x),n}(\cdot) \to Y^{t,x}(\cdot),\ \bar{Y}^{(t,x),n}(\cdot) \to \bar{Y}^{t,x}(\cdot), \text{ a.e. on } [t,T]\times\Omega \text{ and in } \mathbb{L}^2(m;t,T), \tag{4.48}$$

so

$$Y^{t,x}(s) \leq \bar{Y}^{t,x}(s),\ t \leq s \leq T. \tag{4.49}$$

□

Theorem 4.4 *(Continuous Dependence Property)*
Under Assumption 4.1, if $(Y^{t,x}, Z^{t,x}, K^{t,x})$ solves the system (4.1) and (4.2), and $(\bar{Y}^{t,x}, \bar{Z}^{t,x}, \bar{K}^{t,x})$ solves the system (4.1) and

$$\begin{cases} \bar{Y}^{t,x}(s) = \bar{\xi}(X^{t,x}(T)) + \int_s^T g(r, X^{t,x}(r), \bar{Y}^{t,x}(r), \bar{Z}^{t,x}(r))dr - \int_s^T \bar{Z}^{t,x}(r)dB_r \\ \qquad\qquad + \bar{K}^{t,x}(T) - \bar{K}^{t,x}(s); \\ \bar{Y}^{t,x}(s) \geq L(s, X^{t,x}(s)),\ t \leq s \leq T,\ \int_t^T (\bar{Y}^{t,x}(s) - L(s, X^{t,x}(s)))d\bar{K}^{t,x}(s) = 0, \end{cases} \tag{4.50}$$

then

$$\begin{aligned} &\mathbb{E}[(Y^{t,x}(s) - \bar{Y}^{t,x}(s))^2] + \mathbb{E}\left[\int_s^T (Z^{t,x}(r) - \bar{Z}^{t,x}(r))^2 dr\right] \\ \leq &\mathbb{E}[|\xi - \bar{\xi}|^2] + C\mathbb{E}\left[\int_s^T (Y^{t,x}(r) - \bar{Y}^{t,x}(r))^2 dr\right]^{\frac{1}{2}},\ 0 \leq t \leq s \leq T. \end{aligned} \tag{4.51}$$

Proof. Apply Itô's rule to $(Y^{t,x} - \bar{Y}^{t,x})^2$, and integrate from s to T. Use Lemma 4.1 and Assumption 4.1 (2). □

Remark 4.2 *When the driver g is concerned about in Assumption 4.1, 4.1 (2) (linear growth rates in y and z, and polynomial growth rate in x) is crucial in bounding the $\mathbb{L}^2$-norms thus proving convergence of a Lipschitz approximating sequence. It is likely that the continuity Assumption 4.1 (3) can be relaxed, because a measurable function can always be approximated by continuous functions of the same growth rate.*

Remark 4.3 *The results in section 3 and section 4 are valid for any arbitrary filtered probability space that can support a d-dimensional Brownian motion. In particular, in the canonical space set up at the beginning of section 2, we may replace Assumption 4.1 (1) and (2) with the more general Assumption 4.1 (1') and (2'), while still getting exactly the same statements in section 4 with tiny modifications of the proofs. Assumption 4.1 corresponds to Assumption 2.1 on the state process $X(\cdot)$ in (2.3). The growth rate (2.53) of the Hamiltonians (2.51) and (2.52) satisfies Assumption 4.1 (2').*

Assumption 4.1 *(1') In (4.1), the drift $f : [0,T] \times C^l[0,\infty) \to \mathbb{R}^l$, $(t,\omega) \mapsto f(t,\omega(t))$, and volatility $\sigma : [0,T] \times C^l[0,\infty) \to \mathbb{R}^{l\times d}$, $(t,\omega) \mapsto \sigma(t,\omega(t))$, are deterministic, measurable mappings such that*

$$|f(t,\omega(t)) - f(t,\bar{\omega}(t))| + |\sigma(t,\omega(t)) - \sigma(t,\bar{\omega}(t))| \leq C \sup_{0\leq s\leq t} |\omega(s) - \bar{\omega}(s)|, \tag{4.52}$$

and

$$|f(t,\omega(t))|^2 + |\sigma(t,\omega(t))|^2 \leq C\left(1 + \sup_{0\leq s\leq t}|\omega(s)|^2\right), \tag{4.53}$$

with some constant C for all $0 \leq t \leq T$, ω and $\bar{\omega}$ in $C^l[0,\infty)$.
(2') In (4.2), the driver g is a deterministic measurable mapping $g : [0,T] \times C^l[0,\infty) \times \mathbb{R}^m \times^{m\times d} \to \mathbb{R}^m$, $(t,\omega,y,z) \mapsto g(t,\omega(t),y,z)$. And

$$|g(t,\omega(t),y,z)| \leq b\left(1 + \sup_{0\leq s\leq t}|\omega(s)|^p + |y| + |z|\right), \tag{4.54}$$

with some positive constant b for all $(t,\omega,y,z) \in [0,T] \times C^l[0,\infty) \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$.

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