Homework 9

Due: 20 Apr 2016

Problem 1
Prove Lemma 4.8 (properties of outer measure) in the class notes.

Problem 2
Recall that the gamma distribution with parameters $(\alpha, \lambda)$ is the distribution on $(0, \infty)$ with Lebesgue density
\[ p(x) = \Gamma(\lambda)^{-1} \alpha^{\lambda-1} e^{-\alpha x}. \]
Suppose $X$ and $Y$ are independent gamma variables with parameters $(\alpha, \lambda_y)$ and $(\alpha, \lambda_x)$. Show that $X \perp \perp X + Y$ and $X \perp \perp X + Y$.

Problem 3
Many stochastic processes can be interpreted as distributions on functions, i.e. as random variables whose sample space is a space of functions. Suppose we are interested in functions $f: \mathbb{R}^+ \to \mathbb{R}$. The set of all such functions is the product space $\mathbb{R}^{\mathbb{R}^+}$. More generally, the set of all functions from $T$ to $X$ is the product set $X^T$. One of the main technical obstacles in the construction of stochastic processes is that $T$ is often uncountable (such as in the case $T = \mathbb{R}^+$ above); if so, the product set $X^T$ is still well-defined, but the product $\sigma$-algebra and product topology are becoming too coarse. The purpose of this problem, and of problem 4 below, is to understand this phenomenon better.

Let $X$ be a second-countable Hausdorff space with Borel $\sigma$-algebra $\mathcal{B}(X)$. Let $T$ be an arbitrary set, and consider the product space
\[ X^T := \prod_{t \in T} X_t, \]
in which each factor is a copy of $X_t := X$ of $X$. For each $t$, denote by $\text{pr}_t: X^T \to X_t$ the projection map (cf. Definition 2.4 in the class notes). We write $\mathcal{B}(X)^T$ for the product $\sigma$-algebra (the smallest $\sigma$-algebra on $X^T$ which makes all projections measurable), and $\mathcal{B}(X^T)$ for the Borel $\sigma$-algebra generated by the product topology.

Question (a): Show that $\mathcal{B}(X)^T \subset \mathcal{B}(X^T)$.

Question (b): If $T$ is countable, show that equality holds, i.e. $\mathcal{B}(X)^T = \mathcal{B}(X^T)$.

Question (c): Show that $\mathcal{B}(X)^T \neq \mathcal{B}(X^T)$ can hold if $T$ is uncountable. In particular, show that, if $X$ contains more than one element, the singleton set $\{x\}$, for $x \in X^T$, are not in $\mathcal{B}(X)^T$.

Problem 4
Let $T$ be an uncountable set. Let $\Omega$ be a set, and $\mathcal{A}_t$ a $\sigma$-algebra on $\Omega$ for each $t \in T$.

Question (a): Show that, for every $A \in \sigma\left(\bigcup_{t \in T} \mathcal{A}_t\right)$, there exists a countable subset $I \subset T$ such that $A \in \sigma\left(\bigcup_{t \in I} \mathcal{A}_t\right)$.

Question (b): Let $X^T$ again be the product space in Problem 3. Show that an event $A$ is measurable in the product $\sigma$-algebra $\mathcal{B}(X)^T$ if and only if it depends only on a countable number of coordinates. That
is: Show that, whenever $A \in B(X)^T$, there exists a countable subset $I \subset T$ and sets $A_t \in B(X_t)$, for $t \in I$, such that $x \in A$ iff $\text{pr}_t x \in A_t$ for all $t \in A$. 