Homework 4

Due: 24 February 2014

Problem 1 (Warm-up)

Question: Show that every metrizable topological space is a Hausdorff space.

Problem 2 (Metric topologies)

Let $\mathcal{X}$ be a set and $d$ a metric on $\mathcal{X}$. We had defined the metric topology on $\mathcal{X}$ as

$$\tau := \{ A \subseteq \mathcal{X} | \overline{A} \text{ closed with respect to } d \}.$$

Question: Show that the set system $\tau$ above satisfies the properties of a topology, as required in Definition 2.1.

Problem 3 (Distance functions of sets are continuous)

For any subset $A$ of a metric space $(\mathcal{X}, d)$, define the distance function of $A$ as

$$d(x, A) := \inf_{y \in A} d(x, y).$$

Question: Show that $x \mapsto d(x, A)$ is Lipschitz continuous with Lipschitz constant 1.

Problem 4 (Measures on the Borel sets are determined by open sets)

Lemma 2.7 states: If two measures defined on the Borel $\sigma$-algebra of $\mathcal{X}$ coincide on all open sets, or if they coincide on all closed sets, then they are identical.

Question: Please prove this result without assuming the result that $\cap$-stable generators determine measures; instead, use the monotone class theorem.

Problem 5 (Weak convergence of point masses)

Let $P$ be a probability measure on a metrizable space $\mathcal{X}$, and $(x_n)$ a sequence in $\mathcal{X}$ such that $\delta_{x_n} \xrightarrow{w} P$.

Question: Show that $P = \delta_x$ for some $x$.

Problem 6 (Tightness and countable additivity)

Let $\mu$ be a non-negative, finitely additive set function on a measurable space $(\mathcal{X}, \mathcal{A})$, and finite (i.e. $\mu(\mathcal{X}) < \infty$).

Question (a): Show that $\mu$ is a measure (i.e. that it is countably additive) if and only if $\mu(A_n) \searrow 0$ whenever $A_n \searrow \emptyset$. 
A family of sets $C \subset A$ is called a called a **compact class** if every sequence $(C_n)$ of sets in $C$ has the following property:

every finite subset of sets $C_n$ in $(C_n)$ has non-empty intersection $\Rightarrow$ $(C_n)$ has non-empty intersection

The set function $\mu$ is **tight** with respect to a compact class $C$ if

$$\mu(A) = \sup\{\mu(C) | C \in C \text{ and } C \subset A\}.$$  

You will notice the similarity to the definition of inner regularity. Inner regularity is not a form of tightness, however, since the closed sets do not form a compact class. The prototypical compact class are the compact subsets of a Hausdorff space, and the term “tight measure”, without further qualification, is usually used to refer to measures which are tight with respect to the compact sets.

**Question (b):** Show that, if $\mu$ is tight with respect to some compact class $C \subset A$, then it is countably additive on $A$.

**Note:** Question (b) is a bit more difficult than the other problems in this homework.