Homework 2

Due: 10 February 2016

Problem 1
Let \((T, \preceq)\) be a directed set and \(\mathcal{F} = (\mathcal{F}_s)_{s \in T}\) a filtration. For each \(i = 1, \ldots, n\), let \((X^i_s, \mathcal{F}_s)_{s \in T}\) be a martingale.

Question: Show that \((\max_{i \leq n} X^i_s, \mathcal{F}_s)\) is a submartingale.

Problem 2 (Azuma’s inequality)
Let \((X_t, \sigma(X_t))_{t \in \mathbb{N}}\) be a martingale, \((c_t)_{t \in \mathbb{N}}\) be a sequence of non-negative constants, and define \(\mu := \mathbb{E}[X_t]\). (Note \(\mu\) does not depend on \(t\).) The purpose of this problem is to prove Azuma’s inequality, recall: If

\[ |X_{t+1} - X_t| \leq c_{t+1} \quad \text{for all } t \quad \text{and} \quad |X_1 - \mu| \leq c_1, \quad (1) \]

then

\[ \mathbb{P}\{|X_t - \mu| \geq \lambda\} \leq 2 \exp\left(-\frac{\lambda^2}{2 \sum_{s=1}^{t} c_s^2}\right) \quad \text{for all } \lambda > 0. \quad (2) \]

We will use the following general version of Markov’s inequality: For any real-valued random variable \(X\) and any monotonically increasing function \(f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}\),

\[ \mathbb{P}\{|X| \geq \lambda\} \leq \frac{\mathbb{E}[f(|X|)]}{f(\lambda)} \quad \text{for all } \lambda > 0 \text{ with } f(\lambda) > 0. \quad (3) \]

To show the inequality holds, first consider a random variable \(Y\) with values in \([-1, 1]\), and show the following:

Question (a): There is a random variable \(Z\) with values in \([-1, 1]\) such that \(\mathbb{E}[Y|Z] = Z\).

Question (b): If additionally \(\mathbb{E}[Y] = 0\), then \(\mathbb{E}[\exp(\lambda Y)] \leq \cosh(\lambda) \leq \exp(\lambda^2/2)\).

Next, consider the martingale \((X_t)\) and assume the hypothesis (1) holds.

Question (c): Show that

\[ \mathbb{E}[e^{\lambda X_t}] \leq \exp\left(\frac{1}{2} \lambda^2 \sum_{s=1}^{t} c_s^2\right). \quad (4) \]

Question (d): Deduce (2).

Hint: Use Jensen’s inequality in (b). To apply the Markov inequality, use \(f(a) = \exp(ab)\) for a suitable \(b\).

Problem 3 (Potentials)
Let \((X_t, \mathcal{F}_t)\) be a positive, discrete-time supermartingale. (Such a process is sometimes called a potential.)

Question: Show that \(\lim_t \mathbb{E}[X_t] = 0\) implies \(X_t \to 0\) almost surely and in \(L_1\).