Homework 10

(Optional)

Problem 1 (Brownian motion as a Markov process)
Show that Brownian motion has independent increments.

Problem 2 (Mean measures of Poisson processes)
Let $\Pi$ be a Poisson process on $(\mathcal{X}, \mathcal{A}_X)$, as specified in Definition 5.35. Show that the set function $\mu : \mathcal{A}_X \to [0, \infty]$ defined by
$$\mu(A) := \mathbb{E}[\Pi \cap A]$$
for any $A \in \mathcal{A}_X$ is a measure.

Problem 3 (Transformations of Poisson processes)
Deduce Corollary 5.40 from Algorithm 5.38. In other words, let $(\mathcal{X}, \mathcal{A}_X)$ be an uncountable, measurable space such that $\mathcal{X} \times \mathcal{X}$ has measurable diagonal, let $\mu$ and $\nu_1, \nu_2, \ldots$ be measures satisfying (5.74), and $\phi : \mathcal{X} \to \mathcal{X}$ a measurable map.

Question (a): Show that $\phi(\Pi^\mu) =_{a.s.} \Pi^{\phi(\mu)}$ if $\mu$ is $\sigma$-finite.

Question (b): Show that $\Pi^\mu(\cdot) \cap A =_{a.s.} \Pi^{\mu(\cdot) \cap A}$ for any set $A \in \mathcal{A}_X$.

Question (c): Show that $\bigcup_n \Pi^n =_{a.s.} \Pi^{\sum_n \nu_n}$.

In each case, $\Pi^\mu$ denotes the Poisson process with mean measure $\mu$.

Problem 4 (Chapman-Kolmogorov equations)
Let $P_{u_1, \ldots, u_n}$ be the measures constructed in the proof of Theorem 5.24. Show that, if the kernels $p_{su}$ from which the measures are constructed satisfy the Chapman-Kolmogorov equation (5.46), the family
$$\{P_{u_1, \ldots, u_n} \mid u_1 < \ldots < u_n \text{ in } U\}$$
is projective (with respect to product space projection).