Probability Theory II (G6106)

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http://stat.columbia.edu/∼porbanz/G6106S15.html

Peter Orbanz
porbanz@stat.columbia.edu
Morgane Austern
ma3293@columbia.edu

Homework 10

(Optional)

Problem 1 (Brownian motion as a Markov process)

Show that Brownian motion has independent increments.

Problem 2 (Mean measures of Poisson processes)

Let Π be a Poisson process on $(\mathcal{X}, \mathcal{A}_{\mathcal{X}})$, as specified in Definition 5.35. Show that the set function $\mu : \mathcal{A}_{\mathcal{X}} \to [0, \infty]$ defined by

$$\mu(A) := \mathbb{E}\big[|\Pi \cap A|\big] \qquad \text{for any } A \in \mathcal{A}_{\mathcal{X}}$$

is a measure.

Problem 3 (Transformations of Poisson processes)

Deduce Corollary 5.40 from Algorithm 5.38. In other words, let $(\mathcal{X}, \mathcal{A}_{\mathcal{X}})$ be an uncountable, measurable space such that $\mathcal{X} \times \mathcal{X}$ has measurable diagonal, let μ and $\nu_1, \nu_2, \ldots$ be measures satisfying (5.74), and $\phi : \mathcal{X} \to \mathcal{X}$ a measurable map.

Question (a): Show that $\phi(\Pi^{\mu}) =_{\text{a.s.}} \Pi^{\phi(\mu)}$ if μ is σ-finite.

Question (b): Show that $\Pi^{\mu(\bullet)} \cap A =_{\text{a.s.}} \Pi^{\mu(\bullet \cap A)}$ for any set $A \in \mathcal{A}_{\mathcal{X}}$.

Question (c): Show that $\bigcup_n \Pi^{\nu_n} =_{\text{a.s.}} \Pi^{\sum_n \nu_n}$.

In each case, Π^{μ} denotes the Poisson process with mean measure μ.

Problem 4 (Chapman-Kolmogorov equations)

Let $P_{u_1,\ldots,u_n}$ be the measures constructed in the proof of Theorem 5.24. Show that, if the kernels $\mathbf{p}_{su}$ from which the measures are constructed satisfy the Chapman-Kolmogorov equation (5.46), the family

$$\{P_{u_1,\ldots,u_n} \,|\, u_1 < \ldots < u_n \text{ in } U\}$$

is projective (with respect to product space projection).