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# ON THE DENSITY FUNCTIONS OF INTEGRALS OF GAUSSIAN RANDOM FIELDS

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**Abstract**

In the paper, we consider the density functions of random variables that can be written as integrals of exponential functions of Gaussian random fields. In particular, we provide closed form asymptotic bounds for the density functions and under smoothness conditions we derive exact tail approximations of the density functions.

*Keywords:* Gaussian random fields, integral, density function, change of measure

## 1. Introduction

Consider a Gaussian random field $f(t)$ living on a $d$-dimensional compact set $T$. We say that $f(t)$ is a Gaussian random field if for any finite subset $\{t_1, ..., t_n\} \subset T$, $(f(t_1), ..., f(t_n))$ follows a multivariate Gaussian distribution. In this paper, we consider the random variable

$$\log\left(\int_T e^{\sigma(t)f(t)} d\vartheta(t)\right) \tag{1}$$

for some positive function $\sigma(t)$ and a finite measure $\vartheta$. Of interest is the tail behavior of the density function of (1).

The integral of exponential functions of Gaussian random fields plays an important role in both applied probability and statistics. We present a few of them. In spatial point process modeling, let $\lambda(t)$ be the intensity of a Poisson point process on $T$, denoted by $\{N(A) : A \subset T\}$. In order to build in spatial dependence structure, the log-intensity is typically modeled as a Gaussian process, that is, $\log \lambda(t) = f(t) + \mu_f(t)$ and then $E(N(A)|\lambda(\cdot)) = \int_A e^{f(t)+\mu_f(t)} dt$, where $\mu_f(t)$ is the deterministic mean function

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and $f(t)$ is a zero-mean Gaussian process. For instance, [18] considers the time series setting in which $T$ is a one dimensional interval, $\mu_f(t)$ is modeled as the observed covariate process and $f(t)$ is an autoregressive process. See [21, 17, 43, 19, 20] for more examples of such kinds. Under this setting, one can show that $P(N(T) > b) \sim P(\int_T e^{f(t)+\mu_f(t)} dt > b)$ as $b \to \infty$ (see [30]).

In portfolio risk analysis, consider a portfolio of $n$ assets $S_1$,...,$S_n$. The asset prices are usually molded as log-normal random variables. That is, let $X_i = \log S_i$ and further $(X_1, ..., X_n)$ follow a multivariate normal distribution. The total portfolio value $S = \sum_{i=1}^n S_i$ is the sum of dependent log-normal random variables (see [23, 6, 10, 26, 22]). [7] derives the tail asymptotics of $S$ when $n$ is a fixed number. This asymptotic approximation can also be obtained by a more general result in [25]. If one can represent each asset price by a Gaussian random field at one location, that is, $X_i = f(t_i)$, then as the portfolio size becomes large and the asset prices become more correlated, the unit share price of the portfolio admits the limit $\lim_{n\to\infty} S/n = \int e^{f(t)} d\vartheta(t)$. See [13, 30] for detailed discussions on the random field representations of large portfolios.

In option pricing, the asset price (as a function of time) is typically modeled as a geometric Brownian motion ([12, 32]), that is, $S(t) = e^{W(t)}$, where $W(t)$ is a Brownian motion. Then the payoff of an Asian call option with strike price $K$ is $\max(\int_0^T e^{W(t)} dt - K, 0)$.

The literature of extreme behavior of Gaussian random fields focuses mostly on the tail probabilities of $\sup_T f(t)$. The results contain general bounds as well as sharp asymptotic approximations as $b \to \infty$. A partial literature contains [27, 31, 34, 15, 16, 28, 37, 11, 4]. Several methods have been introduced to obtain bounds and asymptotic approximations, each of which imposes different regularity conditions on the random fields. A general upper bound for the tail of $\sup f(x)$ is developed in [15, 40], which is known as the Borel–TIS lemma. For asymptotic results, there are several methods. The double sum method ([33]) requires an expansion of the covariance function around the global maximum of the variance and the mean functions and also locally stationary structures. The Euler–Poincaré Characteristics of the excursion set approximation (denoted by $\chi(A_b)$, where $A_b$ is the excursion set) uses the fact $P(\sup f(x) > b) \approx E(\chi(A_b))$ and requires the random field to be at least twice differentiable ([1, 38, 5, 39]). The tube method ([36]) uses the Karhunen-Loève expansion and imposes differentiability assumptions on the covariance function (fast decaying eigenvalues) and regularity conditions on the random field. The Rice method ([8, 9]) represents the distribution of $\sup f(t)$ (density function) in an implicit form. Recently, [3] studies the geometric properties of high level excursion set for infinitely divisible non-Gaussian fields as well as the conditional distributions of such properties given the high excursion. Bounds of density functions of $\sup f(t)$ have been studied in [41] and [14].

The distribution of the random variable in (1) is studied in the literature when $f(t)$ is a Brownian motion ([42, 24]). Recently, [29] derives the asymptotic approximations of $P(\int_T e^{f(t)}dt > b)$ as $b \to \infty$ for three times differentiable and homogeneous Gaussian random fields. [30] further extends the results to the case when the process has a varying mean function. The density function of (1) for a general Gaussian random field is still unexplored, which is the main target of this paper. The results derived in this paper lead immediately to bounds and approximations of the tail probabilities $P(\int_T e^{\sigma(t)f(t)}d\vartheta(t) > b)$ by integrating the density on $[b, \infty)$. In addition, such a kind of local results provides technical supports of the theoretical analysis of simulation studies, in which one typically needs to simulate a discrete process to approximate the continuous process. As shown in the technical development in [2] (focusing on the simulation of the tail probabilities of $\sup_T f(t)$), to provide bounds on the bias caused by the discretization, one needs local results (bounds of the density functions) of $\sup_T f(t)$.

The contribution of this paper is to develop asymptotic bounds and approximations of the density functions of (1). Our results consist of several theorems. Asymptotic upper bounds are given in Theorems 1 and 2 under different conditions. An exact approximation of the density is given in Theorem 3 when $f(t)$ is three times differentiable. In addition, during the proof of the theorems, a bound of $F'(a)$ for all $a \in R$ is derived (the results in Section 3.1.3).

The basic technique is to use the Karhunen-Loève expansion $f(t) = \sum_{i=1}^{\infty} x_i \phi_i(t)$ by developing bounds for $f_N(t) = \sum_{i=1}^{N} x_i \phi_i(t)$ and sending $N$ to infinity. For $f_N(t)$, we consider it as a function of $(x_1, ..., x_N)$ and develop bounds of the integral on the surface $\{(x_1, ..., x_N) : \log \int e^{\sigma(t)f_N(t)}d\varphi(t) = a\}$ (endowed with a standard Gaussian measure). Part of the analysis technique is inspired by [41] who presents a bound of the density of $\sup_T f(t)$. The current analysis is more complicated in that $H_f$ is not a sublinear function of $f$, which is a crucial condition in the proof of [41]. [36] also uses this representation to derive an approximation of the tail probability of $\sup_T f(t)$. In addition, a change of measure technique is used to derive explicit forms of the bounds and the asymptotic approximations.

The organization of the rest of this paper is as follows. In Section 2, we present the main results. Proofs of the theorems are given in Section 3. An appendix is added containing the technical proofs of lemmas.

## 2. Main results

Consider a Gaussian random field, $f(t)$, living on a $d$-dimensional compact domain $T \subset R^d$. For a finite measure $\vartheta$ on $T$ and a function $\sigma(t) \in (0,\infty)$ satisfying $\sigma_T = \sup_{t\in T}\sigma(t) < \infty$, let

$$H_f \triangleq \log\left(\int_T e^{\sigma(t)f(t)}d\vartheta(t)\right), \qquad F(a) \triangleq P(H_f \le a). \tag{2}$$

Of interest is the probability density function $F'(a)$. To facilitate the discussion, we present a list of conditions that we will refer to in later discussions.

C1 The index domain $T$ is a $d$-dimensional Borel measurable compact subset of $R^d$ with piecewise smooth boundary. The measure $\vartheta$ is positive and $\vartheta(T) = 1$.

C2 The process $f(t)$ is almost surely continuous with zero mean and unit variance.

Furthermore, we impose two types of structures on the covariance function, under each of which we derive more precise bounds or approximations of $F'(a)$.

C3 The variance is constant, i.e., $\sigma(t) \equiv \sigma$. The measure $\vartheta$ has a positive and continuous density function with respect to the Lebesgue measure. The process $f(t)$ is homogeneous. The covariance function is $C(t) = E(f(s)f(s+t))$, which satisfies the following two conditions:

C3a $C(t)$ satisfies the expansion

$$C(t) = C(0) - |t|^\alpha + o\left(|t|^\alpha\right)\ \ as\ t \to 0,\ \text{for}\ \alpha \in (0,2].$$

C3b For each $t \in R^d$, $C(\lambda t)$ is a monotone decreasing function of $\lambda \in R^+$.

C4 The process $f(t)$ is almost surely at least three times continuously differentiable with respect to $t$. The Hessian matrix of $C(t)$ at the origin is $-I$, where $I$ is the $d \times d$ identity matrix.

**Theorem 1.** *Suppose that Conditions C1 and C2 are satisfied. Then, $F'(a)$ exists almost everywhere and*

$$\limsup_{a\to\infty} \sigma_T^2 a^{-1} e^{\frac{a^2}{2\sigma_T^2}} F'(a) \leq 1,$$

*where $\sigma_T = \sup_{t\in T} \sigma(t) < \infty$.*

**Remark 1.** Under conditions C1 and C2 (very weak conditions), Theorem 1 establishes the existence and an asymptotic bound of $F'(a)$. The following simple example implies that without additional assumptions, the bound in Theorem 1 is efficient up to a polynomial term of $a$. Consider a constant field $f(t) \equiv Z$ where $Z \sim N(0,1)$. Let $\sigma(t)$ take a constant value $\sigma$. Then, $F'(a) = \exp(-a^2/(2\sigma^2))/(\sqrt{2\pi}\sigma)$.

Under more regularity conditions, we further improve the bound.

**Theorem 2.** *Suppose that Conditions C1-3 are satisfied. We write*

$$d\vartheta(t) = \frac{e^{\mu_f(t)}}{\int_T e^{\mu_f(s)} ds} dt, \tag{3}$$

*for some continuous function $\mu_f(t)$ on $T$. For each $\epsilon$ and $a$, let $u_\epsilon$ (as a function of $a$) be the solution to the equation*

$$e^{\sigma u_\epsilon} u_\epsilon^{d\epsilon - d/2\alpha} = e^a \int_T e^{\mu_f(t)} dt. \tag{4}$$

*Then, for any $\epsilon \in (0, \frac{1}{2\alpha})$*

$$u_\epsilon^{d\epsilon - \frac{d}{2\alpha} - 1} e^{\frac{u_\epsilon^2}{2}} F'(a) \to 0, \ \text{as } a \to \infty. \tag{5}$$

**Remark 2.** Note that when $a$ is large, the above equation (4) generally has two solutions. One is on the order of $a/\sigma$; the other one is close to zero. We choose the larger solution as our $u_\epsilon$.

In equation (4), if we replace the integral $\int e^{\mu_f(t)} dt$ by 1 (or any other constant), then $u_\epsilon$ will be shifted by approximately a constant. Denote the corresponding solution by $\tilde{u}_\epsilon$. Note that the results hold for all $\epsilon$ sufficiently small. For $u_\epsilon$ large enough, we have $\tilde{u}_\epsilon < u_{\epsilon/2}$. Thus, the bound in (5) holds by replacing $u_\epsilon$ with $\tilde{u}_\epsilon$.

The exact asymptotic approximation of $F'(a)$ can be derived when $f$ is homogenous and three times differentiable (condition C4). The statement of the theorem needs the following notations. Let "$\partial$" denote the gradient and "$\Delta$" denote the Hessian matrix with respect to $t$. The notation "$\partial^2$" is used to denote the vector of second derivatives with respect to $t$, i.e., $\partial^2 f(t)$ is a $d(d+1)/2$ dimensional vector. The difference between $\partial^2 f(t)$ and $\Delta f(t)$ is that $\Delta f(t)$ is a $d \times d$ symmetric matrix whose diagonal and upper triangle consist of elements of $\partial^2 f(t)$.

It is well known that, for each given $t \in T$, $\big(f(t), \partial^2 f(t)\big)$ is a multivariate Gaussian random vector with mean zero and covariance matrix

$$\Gamma = \begin{pmatrix} 1 & \mu_{20} \\ \mu_{02} & \mu_{22} \end{pmatrix} \tag{6}$$

where $\mu_{20}$ is the vector containing the spectral moments of order two and $\mu_{22}$ is the matrix containing the spectral moments of order four. Both $\mu_{20}$ and $\mu_{22}$ are arranged in an appropriate order according to the order of $\partial^2 f(t)$. See standard textbook, for instance, Chapter 5.5 of [5], for more details of $\mu_{20}$ and $\mu_{22}$.

**Theorem 3.** *Suppose that Conditions C1-4 are satisfied (with the expansion in C3a replaced by C4). Let $\vartheta$ be defined as in (3) and $\mu_f(t)$ is three times differentiable. Then the following approximation holds as $a \to \infty$*

$$F'(a) = (1 + o(1))\sigma^{-1}\tilde{u}^d \int_T \exp\left\{-\frac{(\tilde{u} - \mu_f(t)/\sigma)^2}{2}\right\} \cdot C_H(\mu_f, \sigma, t)dt,$$

*where $\tilde{u}$ (as a function of $a$) is the solution to*

$$\left(\frac{2\pi}{\sigma}\right)^{\frac{d}{2}} \tilde{u}^{-\frac{d}{2}} e^{\sigma\tilde{u}} = e^a \cdot \int_T e^{\mu_f(t)}dt,$$

*the function $C_H$ is defined as*

$$\begin{aligned} & C_H(\mu_f, \sigma, t) \\ = \quad & \frac{|\Gamma|^{-\frac{1}{2}}}{(2\pi)^{\frac{(d+1)(d+2)}{4}}} \exp\left\{\frac{\mathbf{1}^T\mu_{22}\mathbf{1} + \sum_i \partial^4_{iiii}C(0)}{8\sigma^2} + \frac{d \cdot \mu_f(t) + Tr(\Delta\mu_f(t))}{2\sigma^2} + \frac{|\partial\mu_f(t)|^2}{\sigma^2}\right\} \\ & \times \int_{z \in R^{d(d+1)/2}} \exp\left\{-\frac{1}{2}\left[\frac{|\mu_{20}\mu_{22}^{-1}z|^2}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} + \left|\mu_{22}^{-1/2}z - \frac{\mu_{22}^{1/2}\mathbf{1}}{2\sigma}\right|^2\right]\right\} dz, \end{aligned}$$

*$\mu_{20}$, $\mu_{02}$, $\mu_{22}$, and $\Gamma$ are defined in (6), and*

$$\mathbf{1} = (\underbrace{1, ..., 1}_{d}, \underbrace{0, ..., 0}_{d(d-1)/2})^\top.$$

**Remark 3.** For Condition $C1$, if $\vartheta(T) \neq 1$, we can always perform the following transformation

$$\log \int_T e^{\sigma f(t)} d\vartheta(t) = \log \left\{ \frac{1}{\vartheta(T)} \int_T e^{\sigma(t) f(t)} d\vartheta(t) \right\} + \log \vartheta(T)$$

and let $\vartheta'(\cdot) = \vartheta(\cdot)/\vartheta(T)$.

Condition C2 assumes the zero expectation function. For any continuous function $\mu(t)$,

$$\log \int_T e^{\sigma(t) f(t) + \mu(t)} d\vartheta(t) = \log \int_T e^{\sigma(t) f(t)} d\vartheta'(t),$$

where $d\vartheta'(t) = e^{\mu(t)} d\vartheta(t)$. Therefore, this problem setting includes the situation when the mean is not a constant.

Condition C4 implies that $C(t)$ is at least 6 times differentiable and the first, third, and fifth derivatives at the origin are all zero. The assumption that the Hessian matrix is identity is introduced to simplify the notations. For any Gaussian process $g(t)$ with covariance function $C_g(t)$ and $\Delta C_g(0) = -\Sigma$ and $\det(\Sigma) > 0$, this assumption can be obtained by an affine transformation by letting $g(t) = f(\Sigma^{1/2} t)$ and

$$\log \int_T e^{\sigma g(t) + \mu_f(t)} dt = \log \det(\Sigma^{-1/2}) + \log \int_{\{s : \Sigma^{-1/2} s \in T\}} e^{\sigma f(s) + \mu_f(\Sigma^{-1/2} s)} ds,$$

where for each positive semi-definite matrix $\Sigma$ we let $\Sigma^{1/2}$ be a symmetric matrix such that $\Sigma^{1/2} \Sigma^{1/2} = \Sigma$.

## 3. Proof

In this section, we present the proofs of the theorems. We organize the proofs as follows. In Section 3.1, we develop a proposition that is central to the proofs of all the three theorems. The theorems are proved in Section 3.2 based on the results in Section 3.1. To smooth the discussion, we present the statements of lemmas where they are applied and leave their proofs in the appendix.

Throughout the discussion we use the following notations for the asymptotic behaviors. We say that $0 \le g(a) = O(h(a))$ if $g(a) \le c h(a)$ for some constant $c \in (0, \infty)$ and all $a \ge a_0 > 0$; similarly, $g(a) = o(h(a))$ if $g(a)/h(a) \to 0$ as $a \to \infty$.

### 3.1. A general bound for $F'(a)$

**Proposition 1.** *Under the conditions of Theorem 1, $F'(a)$ exists almost everywhere. Choose $b < a$ (depending on $a$) in a way that $a - b \to 0$ and $a(a-b) \to \infty$ when we send $a$ to infinity. Then,*

$$\limsup_{a\to\infty} \sqrt{2\pi}\sigma_T \exp\left(\frac{\sigma_T^2 t_b^2 + 2(a-b)b}{2\sigma_T^2}\right) F'(a) \le 1, \tag{7}$$

*where $\sigma_T = \sup_{t\in T}\sigma(t)$, $t_b = \Phi^{-1}(F(b))$, and $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian distribution.*

We spend the rest of this subsection to prove this proposition. According to the Karhunen-Loève representation theorem (see Chapter 3 in [5]), $f(t)$ has the following expression

$$f(t) = \sum_{i=1}^{\infty} x_i \phi_i(t), \tag{8}$$

where $\{x_i, i \in \mathbf{N}\}$ are i.i.d. standard Gaussian random variables and $\sum_i \phi_i(t)^2 = 1$. For any positive integer $N$, let $f_N(t)$ be the partial sum of the first $N$ terms. Note that $f_N(t)$ can be viewed as a function of $(x_1, ..., x_N)$. We slightly abuse the notations and write

$$f_N(x,t) = \sum_{i=1}^{N} x_i \phi_i(t) \tag{9}$$

where $x \triangleq (x_1, \cdots, x_N)$. When writing $f_N(t)$ we consider it as a random function; when writing $f_N(x,t)$ or $f_N(x,\cdot)$ we emphasize that it is a function of $x$ mapping from $R^N$ to $C(T)$. Similarly, we redefine function $H_{f_N} : R^N \to R$ as

$$H_{f_N}(\cdot) : x \longmapsto H_{f_N}(x) = \log\left[\int_T e^{\sigma(t) f_N(x,t)} d\vartheta(t)\right].$$

Let $\mu_N$ be the standard Gaussian measure on the probability space $(R^N, \mathcal{B}(R^N), \mu_N)$ with density function

$$\varphi_N(x) = (2\pi)^{-N/2} \exp\left(-\frac{1}{2}|x|^2\right), \tag{10}$$

that is, $\mu_N(A) = \int_A \varphi_N(x)dx$, where $|\cdot|$ is the Euclidean distance.

We first establish a bound for the density of $f_N(t)$ and then send $N$ to infinity. On the probability space $(R^N, \mathcal{B}(R^N), \mu_N)$, define the following sets

$$V_{N,a} \triangleq \left\{x \in R^N : H_{f_N}(x) \le a\right\}\ ,\ W_{N,a} \triangleq \left\{x \in R^N : \sup_{t\in T}\{\sigma(t) f_N(x,t)\} \le a\right\}. \tag{11}$$

and distribution functions

$$F_N(a) \triangleq P\left(H_{f_N} \le a\right) = \mu_N(V_{N,a}),$$
$$G_N(a) \triangleq P\left(\sup_{t\in T}\{\sigma(t) f_N(t)\} \le a\right) = \mu_N(W_{N,a}). \tag{12}$$

We prove Proposition 1 in four steps. In Steps 1 and 2, we derive a "not-so-friendly" bound for $F'_N(a)$. In Step 3, we send $N$ to infinity and develop the corresponding bound for $F'(a)$. Finally inequality (7) is proved in Step 4 based on the results in Step 3.

3.1.1. **Step 1.** Let $\nabla H_{f_N}(x)$ be the gradient field of $H_{f_N}(x)$ with respect to $x$ and denote

$$l_x = \frac{1}{|\nabla H_{f_N}(x)|}.$$

Further let $S_a$ be the surface on which $H_{f_N}(x) = a$, i.e.,

$$S_a = \{x : H_{f_N}(x) = a\}.$$

We write

$$\tilde f_N(x,t) = \sigma(t)\cdot f_N(x,t).$$

For $a \in R$, the density function $F'_N(a)$ can be written as a surface integral as follows:

$$F'_N(a) = \lim_{\epsilon\to 0}\frac{F_N(a+\epsilon) - F_N(a)}{\epsilon} = \lim_{\epsilon\to 0}\frac{\mu_N(V_{a+\epsilon}) - \mu_N(V_a)}{\epsilon} = \int_{S_a} l_x \varphi_N(x) dS_a(x), \tag{13}$$

where $\varphi_N$ is defined as in (10) and $dS_a(x)$ denotes the surface integral element on $S_a \subset R^N$.

The next lemma gives a basic inequality that bounds the surface integral by an integral on the set $V_{N,a}$. Its proof follows a similar derivation in [41].

**Lemma 1.** *Consider the probability space $(R^N, \mathcal{B}(R^N), \mu_N)$. Under the conditions in Theorem 1, we have the following bound*

$$\int_{S_a} l_x\varphi_N(x)dS_a(x) \le \int_{V^c_{N,a}} l_{h(x)}\left(c^+_{h(x)}+1\right) d\mu_N(x), \tag{14}$$

*where $V^c_{N,a} = R^N \setminus V_{N,a}$, $h(x) = \arg\min_{z\in S_a}|x-z|$ is the projection of $x$ onto the surface $S_a$,*

$$c_x = \langle x, \mathbf{n}_x\rangle, \qquad c^+_x = \max\{c_x, 0\},$$

*$\langle\cdot,\cdot\rangle$ is the inner product, and $\mathbf{n}_x$ is the unit vector orthogonal to the surface $S_a$ pointing towards the side where $H_{f_N}(x)$ has larger values.*

3.1.2. **Step 2.** We start with deriving bounds for $l_{h(x)}$ and $c_{h(x)}$, where $h(x)$ is defined as in Lemma 1. Note that

$$\partial_i H_{f_N}(h(x)) = e^{-a}\int_T \sigma(t)\phi_i(t)e^{\tilde f_N(h(x),t)}d\vartheta(t)$$

and since $h(x)\in S_a$

$$\begin{aligned} l_{h(x)}^{-1}c_{h(x)} &= \langle h(x), \nabla H_{f_N}(h(x))\rangle \\ &= e^{-a}\int_T \tilde f_N(h(x),t)e^{\tilde f_N(h(x),t)}d\vartheta(t) \\ &\le \sup_{t\in T}\{\tilde f_N(h(x),t)\}\cdot e^{-a}\int_T e^{\tilde f_N(h(x),t)}d\vartheta(t) \\ &= \sup_{t\in T}\{\tilde f_N(h(x),t)\}. \end{aligned} \tag{15}$$

This implies that

$$c_{h(x)}^{+} \le l_{h(x)}\left(\sup_{t\in T}\{\tilde f_N(h(x),t)\}\right)^{+}. \tag{16}$$

The following two lemmas provide a bound for $l_{h(x)}$.

**Lemma 2.** *$V_{N,a}$ is a convex set and $H_{f_N}: R^N \to R$ is a convex function.*

**Lemma 3.** *For each $b<a$ and $x\in S_a$*

$$l_x = |\nabla H_{f_N}(x)|^{-1} \le \frac{\rho(x,V_{N,b})}{a-b},$$

*where $V_{N,b}=\{z\in R^N: H_{f_N}(z)\le b\}$ and $\rho(x,V_{N,b})=\inf_{z\in V_{N,b}}|x-z|$.*

According to (16) and Lemma 3, for each $x\in V_{N,a}^c$, the integrant in (14) is bounded by

$$l_{h(x)}\left(c_{h(x)}^{+}+1\right) \le \frac{\rho(h(x),V_{N,b})}{a-b}\left(\frac{\rho(h(x),V_{N,b})}{a-b}\left(\sup_{t\in T}\tilde f_N(h(x),t)\right)^{+}+1\right), \tag{17}$$

which implies that

$$\begin{aligned} F_N'(a) &\le \int_{V_{N,a}^c} l_{h(x)}\left(c_{h(x)}^{+}+1\right)d\mu_N(x) \\ &\le \int_{V_{N,a}^c}\frac{\rho(h(x),V_{N,b})}{a-b}\left(\frac{\rho(h(x),V_{N,b})}{a-b}\left(\sup_{t\in T}\tilde f_N(h(x),t)\right)^{+}+1\right)d\mu_N(x), \end{aligned} \tag{18}$$

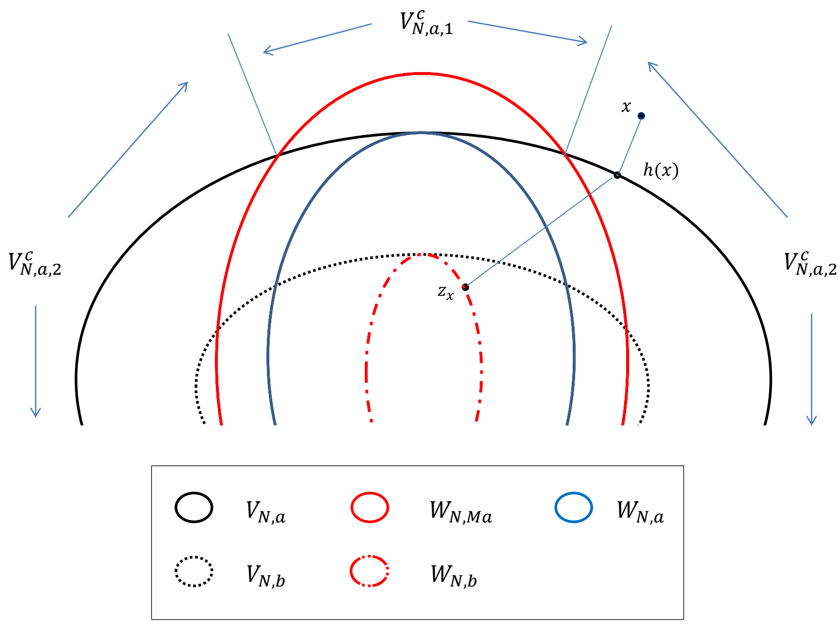


FIGURE 1: This graph illustrates the relative positions of different sets used in the proof. The legends indicate the boundary of each set.

By the fact that for any $x$

$$\log\left[\int_{t\in T} e^{\tilde f_N(x,t)} d\vartheta(t)\right] \le \sup_{t\in T} \tilde f_N(x,t),$$

we obtain $W_{N,a} \subseteq V_{N,a}$ for all $a$, where $W_{N,a}$ is defined as in (11). Now for some constant $M \ge 1$, partition $V^c_{N,a} = R^N \backslash V_{N,a}$ into two parts:

$$V^c_{N,a} = V^c_{N,a,1} \cup V^c_{N,a,2}, \tag{19}$$

where

$$V^c_{N,a,1} = \{x + \lambda \mathbf{n}_x : \lambda \ge 0, x \in S_a,\ \text{and}\ \sup_{t\in T}\{\tilde f_N(x,t)\} < M \cdot a\}$$

and

$$V^c_{N,a,2} = \{x + \lambda \mathbf{n}_x : \lambda \ge 0, x \in S_a,\ \text{and}\ \sup_{t\in T}\{\tilde f_N(x,t)\} \ge M \cdot a\}$$

with $\mathbf{n}_x$ defined as in Lemma 1. Figure 1 illustrates the relative geometric positions of all the relevant sets.

We split the integral (18) into

$$\begin{aligned}F'_N(x) &\le \int_{V^c_{N,a,1}} \frac{\rho(h(x),V_{N,b})}{a-b}\left(\frac{\rho(h(x),V_{N,b})}{a-b}\left(\sup_{t\in T}\tilde f_N(h(x),t)\right)^+ + 1\right) d\mu_N(x)\\ &\quad + \int_{V^c_{N,a,2}} \frac{\rho(h(x),V_{N,b})}{a-b}\left(\frac{\rho(h(x),V_{N,b})}{a-b}\left(\sup_{t\in T}\tilde f_N(h(x),t)\right)^+ + 1\right) d\mu_N(x)\\ &= I_1 + I_2.\end{aligned}$$

We consider the integrals $I_1$ and $I_2$ separately. When $G_N(a) = P(\sup_{t\in T}\tilde f_N(t) \le a) \le 1/2$, we take $M = 1$. Note that in this case, by the fact that $W_{N,a} \subseteq V_{N,a}$, the first term on the right-hand-side of the above display vanishes and we only need to consider the second integral. Then for the first integral we only consider the case that $G_N(a) > 1/2$ (note that $G_N(a) > 1/2$ implies $a > 0$).

**A bound for $I_1$.** By the definition of $V^c_{N,a,1}$, $x \in V^c_{N,a,1}$ implies that $\sup_{t\in T}\tilde f_N(h(x),t) < M\cdot a$ and therefore we have for $a > 0$

$$I_1 \le \int_{V^c_{N,a,1}} \frac{\rho(h(x),V_{N,b})}{a-b}\left(\frac{\rho(h(x),V_{N,b})}{a-b}Ma + 1\right) d\mu_N(x). \tag{20}$$

The following lemma provides a bound for $I_1$.

**Lemma 4.** *For any $a > b$ with $G_N(b) > 1/2$ and increasing function $J(\cdot)$, we have the following inequality:*

$$\int_{V^c_{N,a,1}} J(\rho(x,V_{N,b}))d\mu_N(x) \le \int_{\tau_{M,a,b}+t_{N,b}}^{\infty} J(u - t_{N,b})d\Phi(u),$$

*where*

$$t_{N,b} = \Phi^{-1}(\mu_N(V_{N,b})), \quad \tau_{M,a,b} = \frac{a-b}{Ma}\left(\frac{a-b}{Ma}+1\right)t'_{N,b}$$

*with*

$$t'_{N,b} = \Phi^{-1}\big(P\big(H_{\tilde f_N} < b,\ \sup_{t\in T}\tilde f_N(t) < Ma\big)\big).$$

According to Lemma 4, for any $b < a$ such that $G_N(b) > 1/2$, the right-hand-side of (20) satisfies the following inequality

$$\begin{aligned}&\int_{V^c_{N,a,1}} \frac{\rho(h(x),V_{N,b})}{a-b}\left(\frac{\rho(h(x),V_{N,b})}{a-b}Ma + 1\right) d\mu_N(x)\\ \le\ &\int_{\tau_{M,a,b}+t_{N,b}}^{\infty} \frac{u-t_{N,b}}{a-b}\left(\frac{(u-t_{N,b})}{a-b}Ma + 1\right) d\Phi(u). \end{aligned}\tag{21}$$

This integral can be further bounded by the following inequality whose proof is given in [41].

**Lemma 5.** *For the standard normal distribution, we have the following inequality*

$$\int_{t+r}^{\infty}(u-t)^k d\Phi(u) \quad \leq \quad (1-\Phi(t+r))\, r^k k! \cdot \sum_{i=0}^{k}\frac{1}{(k-i)!r^i(t+r)^i} \text{ for } k\in\mathbf{N}, r>0. \tag{22}$$

Apply Lemma 5 to (21) and obtain that

$$I_1 \quad \leq \quad (1-\Phi(\tau_{M,a,b}+t_{N,b}))\, C_1(a,b,\tau_{M,a,b})$$

where

$$\begin{aligned} & C_1(a,b,\tau_{M,a,b}) \\ = \quad & \frac{Ma}{(a-b)^2}\tau_{M,a,b}^2 + \frac{\tau_{M,a,b}}{a-b} + \frac{2Ma\tau_{M,a,b}+a-b}{(a-b)^2(\tau_{M,a,b}+t_{N,b})} + \frac{2Ma}{(a-b)^2(\tau_{M,a,b}+t_{N,b})^2}. \end{aligned}$$

**A bound for $I_2$.** Choose another constant $\tilde{b}<a$. Given the fact that $W_{N,\tilde{b}}\subseteq V_{N,\tilde{b}}$, we have

$$I_2 \leq \int_{V_{N,a,2}^c}\frac{\rho(x,W_{N,\tilde{b}})}{a-\tilde{b}}\left(\frac{\rho(x,W_{N,\tilde{b}})}{a-\tilde{b}}\left(\sup_{t\in T}\tilde{f}_N(h(x),t)\right)^+ + 1\right)d\mu_N(x). \tag{23}$$

We use the following lemma to further bound $I_2$.

**Lemma 6.** *Consider the probability space $(R^N,\mathcal{B}(R^N),\mu_N)$ and a positive measure set $B$, we have for any increasing function $J$ on $R^+$ and $r>0$*

$$\int_{B_r^c} J(\rho(x,B))d\mu_N(x) \leq \int_{t_B+r}^{\infty} J(u-t_B)d\Phi(u),$$

*where $B_r^c = R^N\backslash B_r = \{x:\rho(x,B)>r\}$, and $t_B \triangleq \Phi^{-1}(\mu_N(B))$.*

In order to use Lemma 6 with $B=W_{N,\tilde{b}}$, we need to derive a lower bound for $\rho(x,W_{N,\tilde{b}})$ for $x\in V_{N,a,2}^c$ (so that each $x\in V_{N,a,2}^c$ is reasonably far away from $W_{N,\tilde{b}}$ and $V_{N,a,2}^c\subseteq B_r^c$) and furthermore an upper bound for $\sup_{t\in T}\tilde{f}_N(h(x),t)$, $h(x)\in S_a$.

Let $\phi^N(t) = (\phi_1(t), \cdots, \phi_N(t))$. For any $z \in W_{N,\tilde{b}}$, any unit-length vector $\mathbf{v}$ and scalar $\lambda$,

$$\begin{aligned} \sup_{t\in T} \tilde{f}_N(z+\lambda\mathbf{v}, t) &= \sup_{t\in T}\left\{\sigma(t)\left\langle z+\lambda\mathbf{v}, \phi^N(t)\right\rangle\right\} \\ &\le \sup_{t\in T}\left\{\sigma(t)\left\langle z, \phi^N(t)\right\rangle\right\} + \sup_{t\in T}\left\{\sigma(t)\left\langle \lambda\mathbf{v}, \phi^N(t)\right\rangle\right\} \\ &\le \tilde{b} + \lambda\sigma_T. \end{aligned} \tag{24}$$

Let $\lambda < (Ma - \tilde{b})/\sigma_T$ and we have that for any unit-length vector $\mathbf{v}$,

$$\sup_{t\in T} \tilde{f}_N\left(z + \frac{Ma-\tilde{b}}{\sigma_T}\mathbf{v}, t\right) < Ma.$$

Thus for any point $x \in R^N$, if $\rho(x, W_{N,\tilde{b}}) < \frac{Ma-\tilde{b}}{\sigma_T}$, then $\sup_t \tilde{f}_N(x,t) < Ma$. Therefore for any $x \in V^c_{N,a,2}$, we have that $\rho(h(x), W_{N,\tilde{b}}) \ge \frac{Ma-\tilde{b}}{\sigma_T}$. Given that $W_{N,\tilde{b}} \subseteq V_{N,a}$ and that $V_{N,a}$ is a convex set, we obtain that $\langle h(x) - y, \mathbf{n}_{h(x)}\rangle > 0$ for all $y \in W_{N,\tilde{b}}$. Thus, we obtain that

$$\rho(x, W_{N,\tilde{b}}) \ge \rho(h(x), W_{N,\tilde{b}}) \ge \frac{Ma-\tilde{b}}{\sigma_T}. \tag{25}$$

Thus, we derived a lower bound of $\rho(x, W_{N,\tilde{b}})$ for $x \in V^c_{N,a,2}$. See Figure 1 for the illustration.

For $x \in V^c_{N,a}$ and $h(x) \in S_a$, let $z_x = \arg\inf_{z\in W_{N,\tilde{b}}} \rho(h(x), z)$ and $\tilde{\mathbf{n}}_{x,z} = (h(x) - z_x)/|h(x) - z_x|$ (see Figure 1). We have an upper bound for $\sup_{t\in T}\tilde{f}_N(h(x),t)$ by the following inequality,

$$\sup_{t\in T}\tilde{f}_N(h(x),t) = \sup_{t\in T}\tilde{f}_N\left(z_x + \rho(h(x), W_{\tilde{b}})\tilde{\mathbf{n}}_{x_z}, t\right) \le \rho(x, W_{\tilde{b}})\sigma_T + \tilde{b},$$

where the last step follows exactly the same argument as in (24). Thus, plugging the above bound for $\sup_{t\in T}\tilde{f}_N(h(x),t)$ into (23), we have

$$\begin{aligned} &\int_{V^c_{N,a,2}} l_{h(x)}\left(c^+_{h(x)}+1\right) d\mu_N(x) \\ \le\ &\int_{V^c_{N,a,2}} \left[\left(\frac{\rho(x, W_{N,\tilde{b}})}{a-\tilde{b}}\right)^2 \left(\rho(x, W_{N,\tilde{b}})\sigma_T + \tilde{b}^+\right) + \frac{\rho(x, W_{N,\tilde{b}})}{a-\tilde{b}}\right] d\mu_N(x). \end{aligned}$$

Then, by Lemma 6 ($B = W_{N,\tilde{b}}$ and $V^c_{N,a,2} \subseteq B^c_{\frac{Ma-\tilde{b}}{\sigma_T}}$), the following inequality holds:

$$\begin{aligned}
&\int_{V^c_{N,a,2}} \left[\left(\frac{\rho(x,W_{N,\tilde{b}})}{a-\tilde{b}}\right)^2 \left(\rho(x,W_{N,\tilde{b}})\sigma_T + \tilde{b}^+\right) + \frac{\rho(x,W_{N,\tilde{b}})}{a-\tilde{b}}\right] d\mu_N(x) \\
\leq\ &\int_{r_{M,a,\tilde{b}}+t_{W_{N,\tilde{b}}}}^{\infty} \left[\left(\frac{u-t_{W_{N,\tilde{b}}}}{a-\tilde{b}}\right)^2 \left((u-t_{W_{N,\tilde{b}}})\sigma_T + \tilde{b}^+\right) + \frac{u-t_{W_{N,\tilde{b}}}}{a-\tilde{b}}\right] d\Phi(u),
\end{aligned}$$

where

$$t_{W_{N,\tilde{b}}} = \Phi^{-1}(G_N(\tilde{b})) \text{ and } r_{M,a,\tilde{b}} = \frac{Ma-\tilde{b}}{\sigma_T}.$$

By Lemma 5 the above integral is bounded by

$$\left(1-\Phi(r_{M,a,\tilde{b}}+t_{W_{N,\tilde{b}}})\right)C_2(a,\tilde{b},r_{M,a,\tilde{b}}),$$

where

$$\begin{aligned}
&C_2(a,\tilde{b},r_{M,a,\tilde{b}}) \\
\triangleq\ &\frac{r_{M,a,\tilde{b}}}{a-\tilde{b}} + \frac{1}{(a-\tilde{b})(t_{W_{N,\tilde{b}}}+r_{M,a,\tilde{b}})} + \frac{\tilde{b}^+}{(a-\tilde{b})^2}\sum_{i=0}^{2}\frac{r^2_{M,a,\tilde{b}}2!}{(2-i)!r^i_{M,a,\tilde{b}}(t_{W_{N,\tilde{b}}}+r_{M,a,\tilde{b}})^i} \\
&+\frac{\sigma_T}{(a-\tilde{b})^2}\sum_{i=0}^{3}\frac{r^3_{M,a,\tilde{b}}3!}{(3-i)!r^i_{M,a,\tilde{b}}(t_{W_{N,\tilde{b}}}+r_{M,a,\tilde{b}})^i}. \qquad (26)
\end{aligned}$$

Combining (23) and (26) together, we have for $a$ such that $G_N(a) > G_N(b) > 1/2$ and $\tilde{b} < a$

$$\begin{aligned}
F'_N(a) \leq \min_{M\geq 1}\Big\{ &\left(1-\Phi(\tau_{M,a,b}+t_{N,b})\right) C_1(a,b,\tau_{M,a,b}) \\
&+\left(1-\Phi(r_{M,a,\tilde{b}}+t_{W_{N,\tilde{b}}})\right)C_2(a,\tilde{b},r_{M,a,\tilde{b}})\Big\}. \qquad (27)
\end{aligned}$$

and for $a$ satisfying $G_N(a) \leq 1/2$, by taking $M = 1$, we have for constant $\tilde{b} < a$

$$\begin{aligned}
F'_N(a) &\leq \int_{V^c_{N,a,2}} \frac{\rho(h(x),V_{N,\tilde{b}})}{a-\tilde{b}} \left(\frac{\rho(h(x),V_{N,\tilde{b}})}{a-\tilde{b}}\left(\sup_{t\in T}\tilde{f}_N(h(x),t)\right)^+ + 1\right) d\mu_N(x) \\
&\leq \left(1-\Phi(r_{a,\tilde{b}}+t_{W_{N,\tilde{b}}})\right)C_2(a,\tilde{b},r_{a,\tilde{b}}),
\end{aligned}$$

with $r_{a,\tilde{b}} = (a-\tilde{b})/\sigma_T$.

3.1.3. **Step 3: Extension to $f(t)$.** From the above derivations, $F_N(a)$ are continuously differentiable on $R$. Let

$$D^+F'_N(a) \triangleq \limsup_{\epsilon\to 0}\frac{F'_N(a+\epsilon)-F'_N(a)}{\epsilon}.$$

By Lemma 11 (presented in the appendix) the total variation of $F_N'$ on any interval $[a_1, a_2]$ satisfies

$$\bigvee_{a_1}^{a_2} F_N' \leq \sup_{a\in[a_1,a_2]} F_N'(a) + 2(a_2 - a_1) \cdot \sup_{a\in[a_1,a_2]} D^+ F_N'(a) \leq m_1 + m_2(a_2 - a_1),$$

for some constants $m_1, m_2 > 0$. Therefore we have that $F_N'(a)$ is continuous on $[a_1, a_2]$ except for a countable set. Also, $F_N'(a)$ is bounded in $L^1$ norm on the interval $[a_1, a_2]$. Then, by Helly's Selection Theorem, there exists a subsequence $\{F_{N_i}'\}_i$ such that it converges almost everywhere (and also in the $L^1$ norm) to a function $\tilde{F}'$ of bounded total variation on $[a_1, a_2]$. Note that $F_N(a)$ converges uniformly to $F(a)$ on interval $[a_1, a_2]$ (Theorem 3.1.2 in [5]). Therefore,

$$F(a_2) - F(a_1) = \lim_i F_{N_i}(a_2) - F_{N_i}(a_1) = \lim_i \int_{a_1}^{a_2} F_{N_i}'(b)db = \int_{a_1}^{a_2} \tilde{F}'(b)db,$$

which implies $\tilde{F}' = F'$ almost everywhere on $[a_1, a_2]$.

Therefore, by the convergence result, we obtain an upper bound of $F'(a)$ by sending $N$ to infinity on both sides of (27), i.e., for $G(a) > G(b) > 1/2$ and $\tilde{b} < a$ (where $G(a) = P(\sup_t \tilde{f}(t) \leq a)$) we have

$$\begin{aligned} F'(a) \quad \leq \quad & \min_{M\geq 1}\Big\{ (1 - \Phi(\tau_{M,a,b} + t_b))\, C_1(a, b, \tau_{M,a,b}) \\ & + \big(1 - \Phi(r_{M,a,\tilde{b}} + t_{W_{\tilde{b}}})\big) C_2(a, \tilde{b}, r_{M,a,\tilde{b}})\Big\}, \end{aligned} \tag{28}$$

and for $a$ such that $G(a) \leq 1/2$ and $\tilde{b} < a$ ,

$$F'(a) \quad \leq \quad \big(1 - \Phi(r_{a,\tilde{b}} + t_{W_{\tilde{b}}})\big) C_2(a, \tilde{b}, r_{a,\tilde{b}}),$$

where $t_b = \Phi^{-1}(F(b))$, $t_{W_{\tilde{b}}} = \Phi^{-1}(G(\tilde{b}))$, $r_{M,a,\tilde{b}} = \frac{Ma-\tilde{b}}{\sigma_T}$, $r_{a,\tilde{b}} = \frac{a-\tilde{b}}{\sigma_T}$, and with a slight abuse of notation $\tau_{M,a,b} = \frac{a-b}{Ma}\left(\frac{a-b}{Ma} + 1\right) t_b'$ with $t_b' = \Phi^{-1}(P(H_f < b,\ \sup_{t\in T} f(t) < Ma))$.

3.1.4. **Step 4.** Based on the result in (28), we now prove Proposition 1 in step 4. We first present the Borel-TIS lemma that is proved by [15, 40].

**Lemma 7.** (Borel-TIS.) *Let $f(t)$, $t \in \mathcal{U}$ ($\mathcal{U}$ is a parameter set), be a mean zero Gaussian random field. $f$ is almost surely bounded on $\mathcal{U}$. Then,*

$$E(\sup_{\mathcal{U}} f(t)) < \infty,$$

*and*

$$P(\max_{t\in\mathcal{U}} f(t) - E[\max_{t\in\mathcal{U}} f(t)] \geq b) \leq e^{-\frac{b^2}{2\sigma^2_{\mathcal{U}}}},$$

*where*

$$\sigma^2_{\mathcal{U}} = \max_{t\in\mathcal{U}} Var[f(t)].$$

Based on the Borel-TIS lemma, we have that

$$\liminf_{a\to\infty} \frac{t_a}{a} \geq \lim_{a\to\infty} \frac{t_{W_a}}{a} = \sigma_T^{-1} \text{ and } t_{W_a} - \sigma_T^{-1} a = O(1).$$

Now choose $b = b(a) < a$ such that, as $a \to \infty$, $a - b \to 0$ and $a(a-b) \to \infty$ and

$$M = \sigma_T \left(1 + \frac{C}{a}\right) \frac{t_b}{b} > 1$$

with a constant $C$ big enough (note that $t_b \geq t_{W_b} \geq b/\sigma_T + O(1)$). In addition, let $\tilde{b}$ be a fixed constant. Under the above settings, as $a \to \infty$, we simplify the functions

$$\begin{aligned} C_1(a, b, \tau_{M,a,b}) &= (1 + o(1)) \cdot \frac{{t'_b}^2}{Ma} \\ C_2(a, \tilde{b}, r_{M,a,\tilde{b}}) &= \frac{\sigma_T (Ma - \tilde{b})^3}{(a - \tilde{b})^2 \sigma_T^3} \cdot (1 + o(1)) = \frac{M^3 a}{\sigma_T^2} \cdot (1 + o(1)). \end{aligned}$$

We now show that the second term in (28) is of a smaller order, that is,

$$\big(1 - \Phi(r_{M,a,\tilde{b}} + t_{W_{\tilde{b}}})\big) C_2(a, \tilde{b}, r_{M,a,\tilde{b}}) = o(1) \cdot (1 - \Phi(\tau_{M,a,b} + t_b))\, C_1(a, b, \tau_{M,a,b}). \quad (29)$$

By choosing $\tilde{b}$ as a constant and sending $a$ to infinity, for some $\lambda > 0$, we have that

$$\begin{aligned} & (r_{M,a,\tilde{b}} + t_{W_{\tilde{b}}})^2 - (t_b + \tau_{M,a,b})^2 \\ = \;& \left(\frac{Ma - \tilde{b}}{\sigma_T} + t_{W_{\tilde{b}}}\right)^2 - \left(t_b + \frac{a-b}{Ma}\left(\frac{a-b}{Ma} + 1\right) t'_b\right)^2 \\ \geq \;& (2 + o(1)) a \left(\frac{Ma - \tilde{b}}{\sigma_T} + t_{W_{\tilde{b}}} - t_b - \frac{a-b}{Ma}\left(\frac{a-b}{Ma} + 1\right) t'_b\right) \\ \geq \;& (2 + o(1)) a \left(\frac{Ma}{\sigma_T} - t_b + (t_{W_{\tilde{b}}} - \sigma_T^{-1}\tilde{b}) + o(1)\right) \\ \geq \;& (2 + o(1)) \frac{\lambda}{2\sigma_T} a, \end{aligned}$$

where the second inequality follows from the following argument. By the fact that $t_{W_b} \leq t'_b \leq t_b$, we have

$$\begin{aligned} \frac{a-b}{Ma}\left(\frac{a-b}{Ma} + 1\right) t'_b &= \left(1 + \frac{a-b}{\sigma_T (a + C)\frac{t_b}{b}}\right) \frac{a-b}{\sigma_T (a+C)\frac{t_b}{b}} t'_b \\ &= (1 + o(1)) \frac{a-b}{\sigma_T t_b} t'_b = o(1). \end{aligned} \quad (30)$$

Therefore, we obtain (29) and the second term in (28) is ignorable. Furthermore, by (30), we have that

$$\begin{aligned}(t_b+\tau_{M,a,b})^2 &= \left(t_b+\left(1+\frac{a-b}{\sigma_T(a+C)\frac{t_b}{b}}\right)\frac{a-b}{\sigma_T(a+C)\frac{t_b}{b}}t_b'\right)^2\\ &\geq t_b^2+2\frac{(a-b)b}{\sigma_T(a+C)}t_b'\\ &\geq t_b^2+2\frac{(a-b)b}{\sigma_T^2}+o\,(1)\,,\end{aligned}$$

where the last step follows from the fact that $t_b'\geq t_{W_b}=\sigma_T^{-1}b+O(1)$.

Therefore

$$\begin{aligned}&(1-\Phi(\tau_{M,a,b}+t_b))\,C_1(a,b,\tau_{M,a,b})\\ \leq\;& (1+o(1))\cdot\frac{{t_b'}^2}{Ma}\frac{1}{\sqrt{2\pi}}\frac{1}{\tau_{M,a,b}+t_b}\exp\left(-\frac{t_b^2+2\frac{(a-b)b}{\sigma_T^2}}{2}\right)\\ \leq\;& (1+o(1))\cdot\frac{1}{\sqrt{2\pi}\sigma_T}\exp\left(-\frac{t_b^2+2\frac{(a-b)b}{\sigma_T^2}}{2}\right),\end{aligned}$$

where the last inequality follows from the fact that $t_b'<t_b<t_b+\tau_{M,a,b}$ and $t_b'<Ma/\sigma_T=(a+C)t_b/b$. Thereby, we complete the proof of Proposition 1.

### 3.2. Proof of the theorems

In this section, we prove our theorems based on Proposition 1. We propose a change of measure $Q$ which is central to the proof of our theorems. Let $P$ be the original measure. The probability measure $Q$ is defined such that $P$ and $Q$ are mutually absolutely continuous with the Radon-Nikodym derivative being

$$\frac{dQ}{dP}=\int_T\frac{\exp\left\{-\frac{1}{2}(f(t)-u)^2\right\}}{\exp\left\{-\frac{1}{2}f(t)^2\right\}}d\vartheta(t), \tag{31}$$

for some $u\in R$. Note that $Q$ depends on $u$. To simplify the notations, we omit the index of $u$ in $Q$ when there is no ambiguity. One can verify that (35) is a valid Radon-Nikodym derivative. We will provide further description in Section 3.2.2. See also [30] who used this change of measure to derive asymptotic approximation of $P(\int_T\exp(\sigma f(t)+\mu_f(t))dt>b)$ with $\mu_f(t)$ being a deterministic function.

3.2.1. **Proof of Theorem 1.** In order to use Proposition 1, we first derive a lower bound for $t_a$. For each $u$, we rewrite

$$\frac{dQ}{dP} = \int_T \exp\left(\frac{2uf(t) - u^2}{2}\right) d\vartheta(t). \tag{32}$$

We have

$$\begin{aligned} & P\left(\log \int_T e^{\sigma(t)f(t)} d\vartheta(t) > a\right) \\ = \;& E_Q\left[\frac{dP}{dQ}; \log \int_T e^{\sigma(t)f(t)} d\vartheta(t) > a\right] \\ = \;& e^{\frac{u^2}{2}} \cdot E_Q\left[\frac{1}{\int_T e^{uf(t)} d\vartheta(t)}; \log \int_T e^{\sigma(t)f(t)} d\vartheta(t) > a\right], \end{aligned}$$

where $E_Q$ is the expectation under measure $Q$. Note that

$$\log \int_T e^{\sigma(t)f(t)} d\vartheta(t) > a$$

implies that for $a$ large enough

$$\begin{aligned} \int_T e^{\sigma_T f(t)} d\vartheta(t) \;&\geq\; \int_{T\cap\{f(t)\geq 0\}} e^{\sigma(t)f(t)} d\vartheta(t) \\ &\geq\; e^a - \int_{T\cap\{f(t)<0\}} e^{\sigma(t)f(t)} d\vartheta(t) \\ &\geq\; e^a - 1. \end{aligned}$$

Then by Jensen's inequality, we have conditioning on $\log \int_T e^{\sigma(t)f(t)} d\vartheta(t) > a$ with $a$ large enough,

$$\int_T e^{uf(t)} d\vartheta(t) \;\geq\; [e^a - 1]^{\frac{u}{\sigma_T}} \tag{33}$$

and therefore

$$\begin{aligned} & P\left(\log \int_T e^{\sigma(t)f(t)} d\vartheta(t) > a\right) \\ = \;& e^{\frac{u^2}{2}} \cdot E_Q\left[\frac{1}{\int_T e^{uf(t)} d\vartheta(t)}; \log \int_T e^{\sigma(t)f(t)} d\vartheta(t) > a\right] \\ \leq \;& e^{\frac{u^2}{2}} \cdot [e^a - 1]^{-\frac{u}{\sigma_T}} \\ = \;& \left(1 - e^{-a}\right)^{-\frac{u}{\sigma_T}} \cdot e^{\frac{u^2}{2} - \frac{ua}{\sigma_T}}. \end{aligned}$$

This bound holds for all $u$ and $\exp\left(\frac{u^2}{2} - \frac{ua}{\sigma_T}\right)$ is minimized when $u = a/\sigma$. Thus, for $a$ sufficiently large, the bound of the tail is

$$1 - F(a) = P\left(\log \int_T e^{\sigma(t)f(t)} d\vartheta(t) > a\right) \;\leq\; (1 + o(1)) \exp\left(-\frac{a^2}{2\sigma_T^2}\right). \tag{34}$$

According to the above inequality, we have

$$t_a \geq \Phi^{-1}\left(1-\exp\left(-\frac{a^2}{2\sigma_T^2}\right)\right) = \frac{a}{\sigma_T} - \sigma_T\frac{\log a - \log\sigma_T}{a} + \frac{\tilde{C}}{a} + o\left(\frac{1}{a}\right),$$

where $\tilde{C}$ satisfies $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tilde{C}}{\sigma_T}\right) = 1$. Then by Proposition 1, take $b = a - 1/\sqrt{a}$ and we have

$$\begin{aligned} F'(a) &\leq (1+o(1))\cdot\frac{1}{\sqrt{2\pi}\sigma_T}\exp\left(-\frac{t_b^2 + 2\frac{(a-b)b}{\sigma_T^2}}{2}\right) \\ &\leq (1+o(1))\frac{a}{\sigma_T^2}\cdot\exp\left(-\frac{a^2}{2\sigma_T^2}\right), \end{aligned}$$

which completes our proof.

3.2.2. **Proof of Theorem 2.** Under the assumptions of Theorem 2, $\sigma(t) \equiv \sigma$ and $d\vartheta(t) = \frac{e^{\mu_f(t)}dt}{\int_T e^{\mu_f(t)}dt}$. Then

$$P\left(\log\int_T e^{\sigma f(t)}d\vartheta(t) > a\right) = P\left(\log\int_T e^{\sigma f(t)+\mu_f(t)}dt > a + \log\int_T e^{\mu_f(t)}dt\right).$$

Similar to the proof of Theorem 1, we prove Theorem 2 by deriving an upper bound for

$$P\left(\log\int_T e^{\sigma f(t)+\mu_f(t)}dt > a\right),$$

which helps to get an lower bound for $t_a$ (then replace $a$ by $a + \log\int e^{\mu_f(t)}dt$).

Consider the change of measure:

$$\frac{dQ}{dP} = \frac{1}{mes(T)}\int_T \frac{\exp\left(-\frac{1}{2}(f(t)-u)^2\right)}{\exp\left(-\frac{1}{2}f(t)^2\right)}dt = \frac{1}{mes(T)}\int_T \exp\left(\frac{2uf(t)-u^2}{2}\right)dt, \quad (35)$$

where $mes(T)$ is the Lebesgue measure of $T$. It is more intuitive to describe the measure $Q$ from a simulation point of view ([30]). One can simulate $f(t)$ under the measure $Q$ according to the following two steps:

1. Simulate a random variable $\tau$ uniformly over $T$ with respect to the Lebesgue measure.

2. Given the realized $\tau$, simulate the Gaussian process $f(t)$ with mean $uC(t-\tau)$ and covariance function $C(t)$.

The second step is equivalent to first sampling $f(\tau)$ from $N(u,1)$ and then sampling $\{f(t) : t \neq \tau\}$ from its original conditional distribution under the measure $P$ given $f(\tau)$. It is not hard to verify that the above two-step procedure is consistent with the Radon-Nikodym derivative in (35). Under $Q$, a random variable $\tau$ is first sampled uniformly over $T$, then $f(\tau)$ is simulated with a large mean at level $u$. This implies that the high value of the integral $\int_T e^{\sigma f(t)}dt$ is mostly caused by the fact that the field reaches a high level at one location $t^*$ and such a location $t^*$ is very close to $\tau$. Therefore, the random index $\tau$ localizes the maximum of the field. We can write the tail probability as

$$
\begin{aligned}
& P\left(\log \int_T e^{\sigma f(t)+\mu_f(t)}dt > a\right) \\
= \;& mes(T)e^{u^2/2}E_Q\left[\frac{1}{\int_T e^{uf(t)}dt}; \log \int_T e^{\sigma f(t)+\mu_f(t)}dt > a\right] \\
= \;& e^{u^2/2}\int_T E_Q\left[\frac{1}{\int_T e^{uf(t)}dt}; \log \int_T e^{\sigma f(t)+\mu_f(t)}dt > a\middle|\tau\right] d\tau. \qquad (36)
\end{aligned}
$$

According to Step 2 of the simulation, conditional on $\tau$ and under measure $Q$, the process

$$\bar{f}(t) = f(t) - uC(t-\tau)$$

follows the same law as $f(t)$ under $P$.

Let $u$ be the solution to $e^{a-\sigma u}u^{\frac{d}{2\alpha}-d\gamma} = 1$ with $0 < \gamma < \epsilon$, where $\epsilon$ is chosen as in the theorem statement. Choose $\delta$ such that $e^{-\sup_{t\in T}\mu_f(t)}u^{d/2\alpha+d\gamma} = mes(s \in T : |s| < u^{\delta})$. Keep in mind that $\delta \approx 1/2\alpha + \gamma$. Let

$$\mathcal{L} = \{\sup_{t\in T}\bar{f}(t) < a^{1/2+\eta}\}.$$

For any $\eta$ satisfying $0 < \eta < \alpha\delta - 1/2$, by Jensen's inequality, we have (36) on $\mathcal{L}^c$

$$
\begin{aligned}
& P\left(\log \int_T e^{\sigma f(t)+\mu_f(t)}dt > a, \mathcal{L}^c\right) \\
= \;& mes(T)e^{\frac{u^2}{2}} \cdot E_Q\left[\frac{1}{\int_T e^{uf(t)}dt}; \log \int_T e^{\sigma f(t)}dt > a - \sup_{t\in T}\mu_f(t) \text{ and } \mathcal{L}^c\right] \\
\leq \;& e^{\frac{u^2}{2} - \frac{u(a-\sup_{t\in T}\mu_f(t) - \log mes(T))}{\sigma}} \\
& \times Q\left(\log \int_T e^{\sigma f(t)}dt > a - \sup_{t\in T}\mu_f(t) \text{ and } \sup_{t\in T}\bar{f}(t) > a^{1/2+\eta}\right) \\
= \;& o(1)e^{-\frac{u^2}{2}}, \qquad (37)
\end{aligned}
$$

where the inequality is thanks to (33) and the last step follows from the Borel-TIS Lemma (applied to $\bar{f}$). Therefore we have that

$$\begin{aligned} & P\left(\log\int_T e^{\sigma f(t)+\mu_f(t)}dt > a\right) \\ \leq\quad & e^{u^2/2}\int_T E_Q\left[\frac{1}{\int_T e^{uf(t)}dt}; \log\int_T e^{\sigma f(t)}dt > a - \sup_{t\in T}\mu_f(t) \ and\ \mathcal{L}\Big|\tau\right]d\tau \\ & +o(1)e^{-\frac{u^2}{2}}. \end{aligned} \tag{38}$$

In what follows, we derive an upper bound for the conditional expectation in (38). We first consider the set $\{\log\int_T e^{\sigma f(t)}dt > a - \sup_{t\in T}\mu_f(t)\}$ in (38). Let $\epsilon_u = u^{-\frac{1}{\alpha}+\delta}$ (recall that $\delta$ is some constant such that $e^{-\sup_{t\in T}\mu_f(t)}u^{d/2\alpha+d\gamma} = mes(s\in T : |s| < u^\delta)$). We can write the integral $\log\int_T e^{\sigma f(t)}dt$ into two parts as below:

$$\int_T e^{\sigma f(t)}dt \quad = \quad e^{\sigma u}T_1 + e^{\sigma u}T_2,$$

where

$$T_1 = \int_{|t-\tau|<\epsilon_u} e^{\sigma f(t)}dt = \int_{|t-\tau|<\epsilon_u} e^{\sigma\bar{f}(t)+\sigma u(C(t-\tau)-C(0))}dt$$

and

$$T_2 = \int_{|t-\tau|\geq\epsilon_u} e^{\sigma f(t)}dt = \int_{|t-\tau|\geq\epsilon_u} e^{\sigma\bar{f}(t)+\sigma u(C(t-\tau)-C(0))}dt.$$

Thus, $\log\int e^{\sigma f(t)}dt > a - \sup_{t\in T}\mu_f(t)$ if and only if

$$T_1 + T_2 > e^{-\sup_{t\in T}\mu_f(t)}u^{d\gamma-d/2\alpha}. \tag{39}$$

For $T_1$, since $C(0)-C(t-\tau) = |t-\tau|^\alpha + R(t-\tau)$ where $R(t-\tau) = o(|t-\tau|^\alpha)$

$$\begin{aligned} T_1 &= \int_{|t-\tau|<\epsilon_u} e^{\sigma\bar{f}(t)+\sigma u(C(t-\tau)-C(0))}dt \\ &= \int_{|t-\tau|<\epsilon_u} e^{\sigma\bar{f}(t)-\sigma u(|t-\tau|^\alpha+R(t-\tau))}dt \\ &= u^{-\frac{d}{\alpha}}\int_{|s|<u^\delta} e^{\sigma\bar{f}(\tau+u^{-\frac{1}{\alpha}}s)-\sigma|s|^\alpha-u\sigma R(u^{-\frac{1}{\alpha}}s)}ds. \end{aligned} \tag{40}$$

For $T_2$, by the condition that $\sup_{t\in T\setminus\mathcal{N}_0} C(t) < C(0) = 1$ where $\mathcal{N}_0$ is a neighborhood of 0, we have for $u$ large enough,

$$T_2 \leq e^{-\sigma u^{\delta\alpha}}\int_T e^{\sigma\bar{f}(t)}dt,$$

and on set $\mathcal{L}$, we have

$$T_2 < mes(T) e^{\sigma a^{1/2+\eta} - \sigma u^{\delta\alpha}}. \tag{41}$$

For the term $\int_T e^{uf(t)}dt$ in (38), we have

$$\begin{aligned}
& \int_{|t-\tau|<\epsilon_u} e^{uf(t)}dt \\
= \;& e^{u^2} \cdot \int_{|t-\tau|<\epsilon_u} e^{u\left(\bar{f}(t)+u(C(t-\tau)-C(0))\right)}dt \\
= \;& e^{u^2} u^{-\frac{d}{\alpha}} mes(|s|<u^{\delta}) \cdot \frac{1}{mes(|s|<u^{\delta})} \int_{|s|<u^{\delta}} e^{u\left(\bar{f}(\tau+u^{-\frac{1}{\alpha}}s)-|s|^{\alpha}-uR(u^{-\frac{1}{\alpha}}s)\right)}ds.
\end{aligned} \tag{42}$$

By Jensen's inequality and (39), on the set $\{\log\int e^{\sigma f(t)}dt > a\}$, we have

$$\begin{aligned}
& \frac{1}{mes(|s|<u^{\delta})} \int_{|s|<u^{\delta}} e^{u\left(\bar{f}(\tau+u^{-\frac{1}{\alpha}}s)-|s|^{\alpha}-uR(u^{-\frac{1}{\alpha}}s)\right)}ds \\
\geq \;& \left[\frac{1}{mes(|s|<u^{\delta})} \int_{|s|<u^{\delta}} e^{\sigma\bar{f}(\tau+u^{-\frac{1}{\alpha}}s)-\sigma|s|^{\alpha}-\sigma uR(u^{-\frac{1}{\alpha}}s)}ds\right]^{u/\sigma} \\
= \;& \left(\frac{u^{d/\alpha}T_1}{mes(|s|\leq u^{\delta})}\right)^{u/\sigma} \\
= \;& \left(\frac{u^{d/\alpha}T_1}{e^{-\sup_{t\in T}\mu_f(t)}u^{d/2\alpha+d\gamma}}\right)^{u/\sigma} \\
\geq \;& \left(1-e^{\sup_{t\in T}\mu_f(t)}u^{\frac{d}{2\alpha}-d\gamma}T_2\right)^{u/\sigma} \\
\geq \;& \left(1-e^{\sup_{t\in T}\mu_f(t)}u^{\frac{d}{2\alpha}-d\gamma}mes(T)\exp\left(\sigma a^{1/2+\eta}-\sigma u^{\delta\alpha}\right)\right)^{u/\sigma}.
\end{aligned} \tag{43}$$

The first equality in the above display is due to (40); the second equality is due to the definition of $\delta$; the second inequality is due to (39); the last step is due to (41). Now combining the above results of (40),(41), (42), and (43), we get

$$\begin{aligned}
& E_Q\left[\frac{1}{\int_T e^{uf(t)}dt}; \log\int_T e^{\sigma f(t)}dt > a - \sup_{t\in T}\mu_f(t) \text{ and } \mathcal{L}\Big|\tau\right] \\
\leq \;& E_Q\Bigg[\frac{1}{e^{u^2}e^{-\sup_{t\in T}\mu_f(t)}u^{-\frac{d}{2\alpha}+d\gamma}\left(1-mes(T)e^{\sup_{t\in T}\mu_f(t)}u^{\frac{d}{2\alpha}-d\gamma}e^{\sigma a^{1/2+\eta}-\sigma u^{\delta\alpha}}\right)^{u/\sigma}}; \\
& \qquad u^{\frac{d}{2\alpha}-d\gamma}T_1 > 1-mes(T)e^{\sup_{t\in T}\mu_f(t)}u^{\frac{d}{2\alpha}-d\gamma}e^{\sigma a^{1/2+\eta}-\sigma u^{\delta\alpha}} \text{ and } \mathcal{L}\Bigg] \\
\leq \;& (1+o(1))e^{\sup_{t\in T}\mu_f(t)}u^{\frac{d}{2\alpha}-d\gamma}e^{-u^2}.
\end{aligned} \tag{44}$$

Note that $u$ is the solution to $e^{a-\sigma u}u^{\frac{d}{2\alpha}-d\gamma}=1$. Then following (38) we can obtain that

$$P\left(\log\int_T e^{\sigma f(t)}dt>a\right)\leq (mes(T)+o(1))e^{\sup_{t\in T}\mu_f(t)}u^{\frac{d}{2\alpha}-d\gamma}e^{-\frac{u^2}{2}},$$

which implies that

$$\begin{aligned}1-F(a) &= P\left(\log\int_T e^{\sigma f(t)+\mu_f(t)}dt>a+\log\int_T e^{\mu_f(t)}dt\right)\\ &\leq (mes(T)+o(1))e^{\sup_{t\in T}\mu_f(t)}{u_\gamma}^{\frac{d}{2\alpha}-d\gamma}e^{-\frac{{u_\gamma}^2}{2}},\end{aligned}$$

where $u_\gamma$ is the solution to

$$\int_T e^{\mu_f(t)}dt\cdot e^{a-\sigma u_\gamma}u_\gamma^{\frac{d}{2\alpha}-d\gamma}=1.$$

Then,

$$\begin{aligned}t_a &\geq \Phi^{-1}\left(1-mes(T)e^{\sup_{t\in T}\mu_f(t)}u_\gamma^{\frac{d}{2\alpha}-d\gamma}e^{-\frac{u_\gamma^2}{2}}\right)\\ &= u_\gamma-\frac{\left(\frac{d}{2\alpha}-d\gamma+1\right)\log u_\gamma}{u_\gamma}-\frac{\log(\sqrt{2\pi}mes(T)e^{\sup_{t\in T}\mu_f(t)})}{u_\gamma}+o\left(\frac{1}{a}\right).\end{aligned}$$

Therefore, by Proposition 7, take $b=a-1/\sqrt{a}$ and we have

$$F'(a)\leq (mes(T)+o(1))e^{\sup_{t\in T}\mu_f(t)}\sigma^{-1}u_\gamma^{1+\frac{d}{2\alpha}-d\gamma}e^{-\frac{u_\gamma^2}{2}}.$$

Then for any $\epsilon\in(0,\frac{1}{2\alpha})$, take $\gamma$ such that $\gamma<\epsilon$ and we have

$$\limsup_{a\to\infty}u_\epsilon^{d\epsilon-\frac{d}{2\alpha}-1}e^{\frac{u_\epsilon^2}{2}}F'(a)=0$$

which completes the proof of Theorem 2.

3.2.3. **Proof of Theorem 3.** We cite the following result (see Theorem 3.4 in [30]) that provides an approximation of $F(a)$ for three-time differentiable fields.

**Lemma 8.** *Under the assumptions and notations of Theorem 3,*

$$\begin{aligned}&P\left(\log\int_T e^{\sigma f(t)+\mu_f(t)}dt>a\right)\\ =\ &(1+o(1))u^{d-1}\int_T\exp\left\{-\frac{(u-\mu_f(t)/\sigma)^2}{2}\right\}\cdot C_H(\mu_f,\sigma,t)dt,\end{aligned}$$

*where $u$ is the solution to*

$$\left(\frac{2\pi}{\sigma}\right)^{\frac{d}{2}}u^{-\frac{d}{2}}e^{\sigma u}=e^a.$$

By Lemma 8 we have that for a three times differentiable Gaussian random field satisfying the conditions in Theorem 3,

$$
\begin{aligned}
1 - F(a) &= P\left(\log \int_T e^{\sigma f(t) + \mu_f(t)} dt > a + \log \int_T e^{\mu_f(t)} dt\right) \\
&= (1 + o(1))\tilde{u}^{d-1} \int_T \exp\left\{-\frac{(\tilde{u} - \mu_f(t)/\sigma)^2}{2}\right\} \cdot C_H(\mu_f, \sigma, t) dt, \quad (45)
\end{aligned}
$$

where $\tilde{u}$ is the solution to

$$
\left(\frac{2\pi}{\sigma}\right)^{\frac{d}{2}} \tilde{u}^{-\frac{d}{2}} e^{\sigma \tilde{u}} = e^a \cdot \int_T e^{\mu_f(t)} dt.
$$

Therefore we can get

$$
t_a = \Phi^{-1}\left(\tilde{u}^{d-1} \int_T \exp\left\{-\frac{(\tilde{u} - \mu_f(t)/\sigma)^2}{2}\right\} \cdot C_H(\mu_f, \sigma, t) dt\right) + o\left(\frac{1}{a}\right)
$$

which implies $t_a/a \to \sigma^{-1}$. Then by Proposition 1, let $b = a - 1/\sqrt{a}$ and we have

$$
F'(a) \le (1 + o(1))\sigma^{-2} a (1 - F(a)).
$$

The right-hand-side of the above display is precisely the approximation in the Theorem.

In order to prove the theorem, we need to show that the right-hand-side of the above equality is also an asymptotic lower bound of the density. According to the approximation in (45), we have that

$$
\begin{aligned}
1 - F(a) = \int_a^\infty F'(x) dx &\le (1 + o(1)) \int_a^\infty \sigma^{-2} x (1 - F(x)) dx \\
&= (1 + o(1))(1 - F(a)). \quad (46)
\end{aligned}
$$

We prove the lower bound by reaching a contradiction to (46). If our conclusion does not hold, there exists $\epsilon > 0$ and $\{a_i, i \ge 1\}$ such that $\lim_i a_i \to \infty$ such that

$$
\frac{F'(a_i)}{\sigma^{-2} a_i (1 - F(a_i))} < 1 - \epsilon.
$$

Then

$$
\int_{a_i}^\infty \left[\sigma^{-2} x (1 - F(x)) - F'(x)\right] dx \ge (1 + o(1)) \int_{a_i}^{\tilde{a}_i} \frac{\epsilon}{2} \cdot \sigma^{-2} x (1 - F(x)) dx, \quad (47)
$$

where

$$
\tilde{a}_i = \inf\left\{x > a_i : \sigma^{-2} x (1 - F(x)) - F'(x) > \epsilon/2 \cdot \sigma^{-2} a_i (1 - F(a_i))\right\}.
$$

We have a lower bound for $\tilde{a}_i$ as

$$\tilde{a}_i \geq a_i + \frac{\epsilon/2 \cdot \sigma^{-2} a_i(1-F(a_i))}{\sup_{a\geq a_i} D^+F'(a) + \left|\frac{\partial \sigma^{-2}a(1-F(a))}{\partial a}|_{a=a_i}\right|}$$

Following the result in Lemma 11, we derive an upper bound for $D^+F'(a_i)$ as in the Steps 3 and 4 in last section. Under the conditions of this theorem, we have $M = \sigma(1+C/a)t_a/a \to 1$; then for $b = a - 1/\sqrt{a}$

$$\begin{aligned} D^+F'(a) &\leq \big(1-\Phi(\tau_{M,a,b}+t_b)\big)C_3(a,b,r_{M,a,b}) \\ &\quad +\big(1-\Phi(r_{M,a,b}+t_{W_b})\big)C_4(a,b,r_{M,a,b}) \\ &= (1+o(1))\big(1-\Phi(\tau_{M,a,b}+t_b)\big)C_3(a,b,r_{M,a,b}) \\ &= (1+o(1))\sigma^{-4}a^2\big(1-\Phi(\tau_{M,a,b}+t_b)\big) \\ &= (1+o(1))\sigma^{-4}a^2(1-F(a)). \end{aligned}$$

Therefore

$$\tilde{a}_i \geq a_i + (1+o(1))\frac{\epsilon/2\cdot\sigma^{-2}a_i(1-F(a_i))}{2\sigma^{-4}a_i^2(1-F(a_i))} = a_i + (1+o(1))\frac{\epsilon\sigma^2}{4a_i}.$$

Thus we have for the right side integral in (47)

$$(1+o(1))\int_{a_i}^{\tilde{a}_i} \frac{\epsilon}{2}\cdot\sigma^{-2}x(1-F(x))dx \quad \geq \quad (1+o(1))\eta_\epsilon(1-F(a_i)), \tag{48}$$

where $\eta_\epsilon > 0$ depends on $\epsilon$ and $\sigma$. Then (47) and (48) indicate that for all $a_i$

$$\int_{a_i}^{\infty}\left[\sigma^{-2}x(1-F(x)) - F'(x)\right]dx \geq (1+o(1))\eta_\epsilon(1-F(a_i)).$$

This contradicts the fact (implied by (46)) that

$$\int_a^{\infty}\left[\sigma^{-2}x(1-F(x)) - F'(x)\right]dx = o(1-F(a)).$$

Therefore we complete our proof.

## 4. Appendix: Lemmas in the proofs

In this section, we present the proofs of lemmas used in the previous section.

The following well-known isoperimetric inequality is due independently to [35, 15].

**Lemma 9.** *Let $B$ is a measurable set of positive measure in $R^N$ and*

$$\mu_N(B) = \Phi(a).$$

*Then, we have for every $r \geq 0$,*

$$\mu_N(B_r) \geq \Phi(a + r),$$

*where $B_r = B + rU = \{x + ry : x \in B, y \in U\}$. and $U$ is the unit ball in $R^N$.*

The following result follows from Theorem 1 in [27].

**Lemma 10.** *For any convex set $B$ in $R^n$ and a half space $H = \{x \in R^N : \langle x, \mathbf{n}\rangle \leq a\}$ with some real number $a$ and some unit vector $\mathbf{n}$ such that*

$$\mu_N(B) \geq \mu_N(H) = \Phi(a),$$

*we have for every $r \geq 1$,*

$$\mu_N(rB) \geq \mu_N(rH) = \Phi(ra),$$

*where $rB = \{rx : x \in B\}$.*

We now start proving the lemmas.

*Proof of Lemma 1.* Lemma 1 follows from a similar argument as in [41]. For equation (13), the inequality

$$\exp\left(-\frac{t^2}{2}\right) \leq (t_+ + 1)\int_t^\infty \exp\left(-\frac{u^2}{2}\right) du$$

implies that

$$\begin{aligned}
&\int_{S_a} l_x \varphi_N(x) dS_a(x) \\
= \; & \int_{S_a} l_x \cdot (2\pi)^{-N/2} \exp\left(-\frac{|x|^2 - c_x^2}{2}\right) \exp\left(-\frac{c_x^2}{2}\right) dS_a(x) \\
\leq \; & \int_{S_a} l_x \cdot (2\pi)^{-N/2} \exp\left(-\frac{|x|^2 - c_x^2}{2}\right) (c_x^+ + 1) \int_{c_x}^\infty \exp\left(-\frac{u^2}{2}\right) du dS_a(x) \\
= \; & \int_{S_a} l_x (c_x^+ + 1) \int_0^\infty \varphi_N(x + \lambda \mathbf{n}_x) d\lambda dS_a(x).
\end{aligned}$$

The last step is due to a change of variable $u = c_x + \lambda$ and the fact that

$$|x + \lambda n_x|^2 = |x|^2 + \lambda^2 + 2\lambda c_x.$$

The above surface integral can be bounded by a volume integral,

$$\begin{aligned} & \int_{S_a} l_x(c_x^+ + 1) \int_0^\infty \varphi_N(x + \lambda \mathbf{n}_x) d\lambda dS_a(x) \\ \leq & \int_{S_a} \int_0^\infty l_x(c_x^+ + 1) \prod_{i=1}^{N-1} (1 + \lambda k_i(x)) \varphi_N(x + \lambda \mathbf{n}_x) d\lambda dS_a(x) \\ = & \int_{V_{N,a}^c} l_{h(x)}(c_{h(x)}^+ + 1) \varphi_N(x) dx, \end{aligned}$$

where $k_i(x)$'s are the principle curvatures of $S_a$ at $x$. The above inequality results from the fact that curvatures are nonnegative in that $S_a$ is the border of a convex set. Therefore, we obtain that

$$F_N'(a) \leq \int_{V_{N,a}^c} l_{h(x)} \left( c_{h(x)}^+ + 1 \right) d\mu_N(x).$$

*Proof of Lemma 2.* For any two functions $f$ and $g$, if $\log \left[ \int_T \exp(f(t)) d\vartheta(t) \right] \leq a$ and $\log \left[ \int_T \exp(g(t)) d\vartheta(t) \right] \leq a$ then,

$$\log \left[ \int_T \exp \left( \frac{f(t) + g(t)}{2} \right) d\vartheta(t) \right] \leq \log \left[ \frac{1}{2} \exp(a) + \frac{1}{2} \exp(a) \right] \leq a.$$

Therefore $H_f = \log \left[ \int_T \exp(f(t)) d\vartheta(t) \right]$ is a convex function.

*Proof of Lemma 3.* By Taylor expansion, the norm of the gradient can be defined as

$$|\nabla H_{f_N}(x)| = \sup_{|v'|=1} \lim_{\varepsilon \to 0+} \frac{H_{f_N}(x) - H_{f_N}(x + \varepsilon v')}{\varepsilon}.$$

For each $b < a$, $H_{f_N}(x) = a$, and $x^* = \arg\inf_{z \in V_{N,b}} |x - z|$, let $v^* = (x^* - x)/|x^* - x|$. Then,

$$|\nabla H_{f_N}(x)| \geq \lim_{\varepsilon \to 0+} \frac{H_{f_N}(x) - H_{f_N}(x + \varepsilon v^*)}{\varepsilon}.$$

By convexity of $H$ (Lemma 2) and the fact that $f_N(x + \varepsilon v^*, \cdot)$ is a linear function of $\varepsilon$, $H_{f_N}(x + \varepsilon v^*)$ is a convex function of $\varepsilon$ and therefore

$$\frac{H_{f_N}(x) - H_{f_N}(x + \varepsilon v^*)}{\varepsilon}$$

is a positive and decreasing function of $\varepsilon$. We choose $\varepsilon = |x^* - x| = \rho(x, V_{N,b})$, then $|H_{f_N}(x) - H_{f_N}(x + \varepsilon v^*)| = a - b$. Thus, we obtain a bound

$$|\nabla H_{f_N}(x)| \geq \frac{a-b}{\rho(x, V_{N,b})}.$$

The proof of Lemma 4 needs Lemma 6. Therefore, we prove Lemma 6 first.

*Proof of Lemma 6.* Lemma 9 implies that the following inequality holds:

$$\int_{B^c} J(\rho(x,B))d\mu_N(x) \leq \int_{t_B}^{\infty} J(u - t_B)d\Phi(u).$$

In view of this, we have

$$\begin{aligned}
&\int_{B_r^c} J(\rho(x,B))d\mu_N(x) \\
\leq\ & \int_{B^c} J(\rho(x,B))I(\rho(x,B) \geq r) + J(r)I(\rho(x,B) < r)d\mu_N(x) \\
&- \int_{B_r \setminus B} J(r)I(\rho(x,B) \leq r)d\mu_N(x) \\
\leq\ & \int_{t_B}^{\infty} J(u - t_B)I(u - t_B \geq r) + J(r)I(u - t_B < r)d\Phi(u) - J(r)\left[\mu_N(B_r) - \mu_N(B)\right].
\end{aligned} \tag{49}$$

Note that

$$1 - \mu_N(B_r) = \int_{B^c} I(\rho(x,B) \geq r)d\mu_N(x) \leq \int_{t_B}^{\infty} I(u - t_B \geq r)d\Phi(u) = 1 - \Phi(t_B + r).$$

Then, $\mu_N(B_r) \geq \Phi(t_B + r)$ and further $\mu_N(B_r) - \mu_N(B) \geq \Phi(t_B + r) - \Phi(t_B)$. Insert this result back into (49) and notice that

$$\int_{t_B}^{\infty} J(r)I(u - t_B < r)d\Phi(u) = J(r)[\Phi(t_B + r) - \Phi(t_B)].$$

We then obtain that

$$\int_{B_r^c} J(\rho(x,B))d\mu_N(x) \leq \int_{t_B + r}^{\infty} J(u - t_B)d\Phi(u).$$

*Proof of Lemma 4.* Let

$$a' = \frac{a-b}{Ma}b + b, \text{ and } V'_{N,b} = \{z : H_{f_N}(z) < b \text{ and } \sup_{t \in T} f_N(z,t) < Ma\}.$$

Thanks to the convexity of $V_{N,b}$, $\rho(x, V_{N,b}) \geq \rho(h(x), V_{N,b})$ for all $x \in V_{N,a,1}^c$. We want to apply Lemma 6 by considering $B = V_{N,b}$ and $V_{N,a,1}^c \subset B_{\tau_{M,a,b}}^c$. Thus, we only need to show that for each $x \in S_a \cap \{\sup \tilde{f}_N(x,t) < Ma\}$

$$\rho(x, V_{N,b}) \geq \tau_{M,a,b} = \frac{a-b}{Ma}\frac{a'}{b}t'_{N,b}.$$

By Lemma 3 and inequality (15), we have

$$\rho(x, V_{N,b}) \geq (a-b)l_x \geq \frac{a-b}{Ma}c_x.$$

Therefore we only need to show that

$$c_x \geq \frac{a'}{b}t'_{N,b}.$$

For any $z \in V'_{N,b}$,

$$\begin{aligned} \log \int_T \exp\left(\tilde{f}_N\left(z \cdot \frac{a'}{b}, t\right)\right) dt &= \log \int_T \exp\left(\tilde{f}_N\left(z + z \cdot \frac{a'-b}{b}, t\right)\right) dt \\ &\leq \log \int_T \exp(\tilde{f}_N(z,t))dt + \sup_{t \in T} \tilde{f}_N\left(z \cdot \frac{a'-b}{b}, t\right) \\ &\leq b + \frac{a'-b}{b}Ma = a. \end{aligned}$$

Thus, we have that $\mu_N\left(\left\{z \cdot \frac{a'}{b} : z \in V'_{N,b}\right\}\right) \leq F_N(a)$. Thanks to Lemma 10, we have

$$\Phi\left(t'_{N,b} \cdot \frac{a'}{b}\right) \leq \mu_N\left(\left\{z \cdot \frac{a'}{b} : z \in V'_{N,b}\right\}\right) \leq F_N(a). \tag{50}$$

Consider an $x \in S_a$ and its unit vector $\mathbf{n}_x$ orthogonal to the tangent plane of $S_a$ at x, denoted by $T_x$. According to the convexity of $V_{N,a}$, the entire set of $V_{N,a}$ lies on one side of $T_x$, which is the side opposite of the $\mathbf{n}_x$. The above statement is equivalent to $V_{N,a} \subset \{z : \langle z, \mathbf{n}_x \rangle < c_x\}$ where $c_x = \langle x, \mathbf{n}_x \rangle$. Thus,

$$F_N(a) = \mu_N(V_{N,a}) \leq \mu_N(\{z : \langle z, \mathbf{n}_x \rangle < c_x\}) = \Phi(c_x).$$

Combined with the above inequality with (50), we obtain that for each $x \in S_a \cup V_{N,a,1}^c$, $\Phi(t'_{N,b}a'/b) \leq F_N(a) \leq \Phi(c_x)$ and thus $c_x \geq t'_b a'/b$. Therefore,

$$\rho(x, V_{N,b}) \geq (a-b)l_x \geq \frac{a-b}{Ma}c_x \geq \frac{a-b}{Ma}\frac{a'}{b}t'_{N,b}.$$

which completes our proof by Lemma 6.

The next lemma provides bound for the second derivative of $F(a)$

$$D^+F_N'(a) \triangleq \limsup_{\epsilon\to 0} \frac{F_N'(a+\epsilon)-F_N'(a)}{\epsilon}. \tag{51}$$

**Lemma 11.** *Consider the probability space $(R^N, \mathcal{B}(R^N), \mu_N)$. Under the conditions in Theorem 1, we have when $G_N(a) > 1/2$, for $\tilde{b} < a$ and $b < a$ such that $G_N(b) > 1/2$*

$$\begin{aligned} D^+F_N'(a) \quad \leq \quad & \min_{M\geq 1}\Big\{\big(1-\Phi(\tau_{M,a,b}+t_{N,b})\big)C_3(a,b,\tau_{M,a,b}) \\ & +\big(1-\Phi(r_{M,a,\tilde{b}}+t_{W_{N,\tilde{b}}})\big)C_4(a,\tilde{b},r_{M,a,\tilde{b}})\Big\} \end{aligned}$$

*and for $G_N(a) \leq 1/2$*

$$D^+F_N'(a) \quad \leq \quad \big(1-\Phi(r_{a,\tilde{b}}+t_{W_{N,\tilde{b}}})\big)C_4(a,\tilde{b},r_{a,\tilde{b}}),$$

*where $t_{W_{N,b}} = \Phi^{-1}(G_N(b))$, $r_{M,a,\tilde{b}} = \frac{Ma-\tilde{b}}{\sigma}$, $r_{a,\tilde{b}} = \frac{a-\tilde{b}}{\sigma}$, $\tau_{M,a,b} = \frac{a-b}{Ma}\left(\frac{a-b}{Ma}+1\right)t_{N,b}'$*

$$\begin{aligned} & C_3(a,b,\tau_{M,a,b}) \\ = \quad & \frac{1}{(a-b)^2}\left(\frac{10}{3}\sum_{i=0}^{2}\frac{\tau_{M,a,b}^2 2!}{(2-i)!\tau_{M,a,b}^i(t_{N,b}+\tau_{M,a,b})^i} + \frac{a^2M^2}{(a-b)^2}\sum_{i=0}^{4}\frac{\tau_{M,a,b}^4 4!}{(4-i)!\tau_{M,a,b}^i(t_{N,b}+\tau_{M,a,b})^i}\right). \end{aligned}$$

$$\begin{aligned} & C_4(a,\tilde{b},r_{M,a,\tilde{b}}) \\ = \quad & \frac{10}{3(a-\tilde{b})^2}\sum_{i=0}^{2}\frac{r_{M,a,\tilde{b}}^2 2!}{(2-i)!r_{M,a,\tilde{b}}^i(t_{W_{N,\tilde{b}}}+r_{M,a,\tilde{b}})^i} + \frac{\tilde{b}^2}{(a-\tilde{b})^4}\sum_{i=0}^{4}\frac{r_{M,a,\tilde{b}}^4 4!}{(4-i)!r_{M,a,\tilde{b}}^i(t_{W_{N,\tilde{b}}}+r_{M,a,\tilde{b}})^i} \\ & +\frac{2\sigma\tilde{b}}{(a-\tilde{b})^4}\sum_{i=0}^{5}\frac{r_{M,a,\tilde{b}}^5 5!}{(5-i)!r_{M,a,\tilde{b}}^i(t_{W_{N,\tilde{b}}}+r_{M,a,\tilde{b}})^i} + \frac{\sigma^2}{(a-\tilde{b})^4}\sum_{i=0}^{6}\frac{r_{M,a,\tilde{b}}^6 6!}{(6-i)!r_{M,a,\tilde{b}}^i(t_{W_{N,\tilde{b}}}+r_{M,a,\tilde{b}})^i}. \end{aligned}$$

*Proof of Lemma 11.* As in Step 1 in Section 3.1.1, we have a volume integral bound for $F_N'(a)$

$$D^+F_N'(a) \quad \leq \quad \int_{V_{N,a}^c} l_{h(x)}^2\left((c_x^+)^2+\frac{10}{3}\right)d\mu_N(x).$$

The proof the above bound follows an argument in [41] (in particular, pages 850-851 of that volume) and therefore is omitted.

Similarly to the proof in Section 3, consider the partition of $V_{N,a}^c$: $V_{N,a}^c = V_{N,a,1}^c \cup V_{N,a,2}^c$ for $M \geq 1$ and we have

$$D^+F_N'(a) \quad \leq \quad \int_{V_{N,a,1}^c} l_{h(x)}^2\left((c_x^+)^2+\frac{10}{3}\right)d\mu_N(x) + \int_{V_{N,a,2}^c} l_{h(x)}^2\left((c_x^+)^2+\frac{10}{3}\right)d\mu_N(x). \tag{52}$$

Similarly as in the above derivation for (20), we have for $a, b$ such that $G_N(a) > G_N(b) > 1/2$

$$\begin{aligned}
& \int_{V^c_{N,a,1}} l^2_{h(x)} \left((c^+_x)^2 + \frac{10}{3}\right) d\mu_N(x) \\
\leq & \int_{V^c_{N,a,1}} \frac{\rho(h(x), V_{N,b})^2}{(a-b)^2} \left(\frac{\rho(h(x), V_{N,b})^2}{(a-b)^2} \big(\sup_{t\in T} f_N(h(x),t)\big)^2 + \frac{10}{3}\right) d\mu_N(x) \\
\leq & \int_{\tau_{M,a,b}+t_{N,b}}^{\infty} \frac{(u-t_{N,b})^2}{(a-b)^2} \left(\frac{(u-t_{N,b})^2}{(a-b)^2} \big(Ma\big)^2 + \frac{10}{3}\right) d\Phi(u) \\
= & \big(1-\Phi(\tau_{M,a,b}+t_{N,b})\big) C_3(a,b,r_{M,a,b}). \qquad (53)
\end{aligned}$$

Similarly as in the above derivation for (26), we have for $\tilde{b} < a$,

$$\begin{aligned}
& \int_{V^c_{N,a,2}} l^2_{h(x)} \left((c^+_x)^2 + \frac{10}{3}\right) d\mu_N(x) \\
\leq & \int_{V^c_{N,a,2}} \frac{\rho(h(x), V_{N,\tilde{b}})^2}{(a-\tilde{b})^2} \left(\frac{\rho(h(x), V_{N,\tilde{b}})^2}{(a-\tilde{b})^2} \big(\sup_{t\in T} f_N(h(x),t)\big)^2 + \frac{10}{3}\right) d\mu_N(x) \\
\leq & \int_{r_{M,a,\tilde{b}}+t_{W_{N,\tilde{b}}}}^{\infty} \frac{(u-t_{W_{N,\tilde{b}}})^2}{(a-\tilde{b})^2} \left(\frac{(u-t_{W_{N,\tilde{b}}})^2}{(a-\tilde{b})^2} \big((u-t_{W_{N,\tilde{b}}})\sigma + \tilde{b}\big)^2 + \frac{10}{3}\right) d\Phi(u) \\
\leq & \big(1-\Phi(r_{M,a,\tilde{b}}+t_{W_{N,\tilde{b}}})\big) C_4(a,\tilde{b},r_{M,a,\tilde{b}}), \qquad (54)
\end{aligned}$$

where $t_{W_{N,\tilde{b}}} = \Phi^{-1}(G_N(\tilde{b}))$, $r_{M,a,\tilde{b}} = \frac{Ma-\tilde{b}}{\sigma}$, and the last inequality follows from Lemma 5. Note that when $G_N(a) < 1/2$ we take $M = 1$ and our conclusion holds.

## References


[1] R.J. Adler. *The Geometry of Random Fields.* Wiley, Chichester, U.K.; New York, U.S.A., 1981.

[2] R.J. Adler, J.H. Blanchet, and J.C. Liu. Efficient monte carlo for large excursions of gaussian random fields. *Annals of Applied Probability*, to appear.

[3] R.J. Adler, G. Samorodnitsky, and J.E. Taylor. High level excursion set geometry for non-gaussian infinitely divisible random fields. *preprint*, 2009.

[4] R.J. Adler, G Samorodnitsky, and J.E. Taylor. Excursion sets of three classes of stable random fields. *Advances in Applied Probability*, 42(2):293–318, 2010.

[5] R.J. Adler and J.E. Taylor. *Random fields and geometry.* Springer, 2007.

[6] S.M. Ahsan. Portfolio selection in a lognormal securities market. *Zeitschrift Fur Nationalokonomie-Journal of Economics*, 38(1-2):105–118, 1978.

[7] S. Asmussen and L. Rojas-Nandayapa. Asymptotics of sums of lognormal random variables with Gaussian copula. *Statistics & Probability Letters*, 78:2709–2714, 2008.

[8] J. M. Azais and M. Wschebor. A general expression for the distribution of the maximum of a gaussian field and the approximation of the tail. *Stochastic Processes and Their Applications*, 118(7):1190–1218, 2008.

[9] J. M. Azais and M. Wschebor. *Level sets and extrema of random processes and fields*. Wiley, Hoboken, N.J., 2009.

[10] S. Basak and A. Shapiro. Value-at-risk-based risk management: Optimal policies and asset prices. *Review of Financial Studies*, 14(2):371–405, 2001.

[11] S. M. Berman. An asymptotic formula for the distribution of the maximum of a gaussian process with stationary increments. *Journal of Applied Probability*, 22(2):454–460, 1985.

[12] F. Black and M. Scholes. Pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[13] J. H. Blanchet, J. Liu, and X. Yang. Monte carlo for large credit portfolios with potentially high correlations. *Proceedings of the 2010 Winter Simulation Conference*, 2010.

[14] V.I. Bogachev. *Gaussian Measures*. Mathematical Surveys and Monographs Volume 62. American Mathematical Society, 1998.

[15] C. Borell. The Brunn-Minkowski inequality in Gauss space. *Inventiones Mathematicae*, 1975.

[16] C. Borell. The Ehrhard inequality. *Comptes Rendus Mathematique*, 337(10):663–666, 2003.

[17] M. J. Campbell. Time-series regression for counts - an investigation into the relationship between sudden-infant-death-syndrome and environmental-temperature. *Journal of the Royal Statistical Society Series a-Statistics in Society*, 157:191–208, 1994.

[18] K. S. Chan and J. Ledolter. Monte-carlo em estimation for time-series models involving counts. *Journal of the American Statistical Association*, 90(429):242–252, 1995.

[19] D. R. Cox. Some statistical methods connected with series of events. *Journal of the Royal Statistical Society Series B-Statistical Methodology*, 17(2):129–164, 1955.

[20] D. R. Cox and Valerie Isham. *Point processes*. Monographs on applied probability and statistics. Chapman and Hall, London ; New York, 1980.

[21] R. A. Davis, W. T. M. Dunsmuir, and Y. Wang. On autocorrelation in a poisson regression model. *Biometrika*, 87(3):491–505, 2000.

[22] H. P. Deutsch. *Derivatives and internal models*. Finance and capital markets. Palgrave Macmillan, Houndmills, Basingstoke, Hampshire ; New York, N.Y., 3rd edition, 2004.

[23] D. Duffie and J. Pan. An overview of value at risk. *The Journal of Derivatives*, 4(3):7–49, 1997.

[24] D. Dufresne. The integral of geometric brownian motion. *Advances in Applied Probability*, 33(1):223–241, 2001.

[25] S. Foss and A. Richards. On sums of conditionally independent subexponential random variables. *Mathematics of Operations Research*, 35:102–119, 2010.

[26] P. Glasserman, P. Heidelberger, and P. Shahabuddin. Variance reduction techniques for estimating value-at-risk. *Management Science*, 46(10):1349–1364, 2000.

[27] H. J. Landau and L. A. Shepp. Supremum of a gaussian process. *Sankhya-the Indian Journal of Statistics Series A*, 32(Dec):369–378, 1970.

[28] M. Ledoux and M. Talagrand. Probability in banach spaces : isoperimetry and processes. 1991.

[29] J. Liu. Tail approximations of integrals of gaussian random fields. *Annals of Probability*, to appear, 2011.

[30] Jingchen Liu and Gongjun Xu. Some asymptotic results of gaussian random fields with varying mean functions and the associated processes. *Annals of Statistis*, to appear.

[31] M. B. Marcus and L. A. Shepp. Continuity of gaussian processes. *Transactions of the American Mathematical Society*, 151(2), 1970.

[32] R. Merton. The theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.

[33] V. I. Piterbarg. *Asymptotic methods in the theory of Gaussian processes and fields.* American Mathematical Society, Providence, R.I., 1996.

[34] V.N. Sudakov and B.S. Tsirelson. Extremal properties of half spaces for spherically invariant measures. *Zap. Nauchn. Sem. LOMI*, 45:75–82, 1974.

[35] V.N. Sudakov and V. S. Tsirel'son. Extremal properties of half-spaces for spherically invariant measures. *ZapiskiNauchn.Seminarov LOMI*, 41:14–24, 1974.

[36] J. Y. Sun. Tail probabilities of the maxima of gaussian random-fields. *Annals of Probability*, 21(1):34–71, 1993.

[37] M. Talagrand. Majorizing measures: The generic chaining. *Annals of Probability*, 24(3):1049–1103, 1996.

[38] J. Taylor, A. Takemura, and R. J. Adler. Validity of the expected euler characteristic heuristic. *Annals of Probability*, 33(4):1362–1396, 2005.

[39] J.E. Taylor and R.J. Adler. Euler characteristics for gaussian fields on manifolds. *The Annals of Probability*, 2003.

[40] B.S. Tsirelson, I.A. Ibragimov, and V.N. Sudakov. Norms of Gaussian sample functions. *Proceedings of the Third Japan-USSR Symposium on Probability Theory (Tashkent, 1975)*, 550:20–41, 1976.

[41] V. S. Tsirel'son. The density of the distribution of the maximum of a gaussian process. *Theory Probab.*, 20:847–856, 1975.

[42] M. Yor. On some exponential functionals of brownian-motion. *Advances in Applied Probability*, 24(3):509–531, 1992.

[43] S. L. Zeger. A regression-model for time-series of counts. *Biometrika*, 75(4):621–629, 1988.