MATHEMATICS OF OPERATIONS RESEARCH
Vol. 00, No. 0, Xxxxx 0000, pp. 000–000
ISSN 0364-765X | EISSN 1526-5471 | 00 | 0000 | 0001

INFORMS
DOI 10.1287/xxxx.0000.0000

On the Tail Probabilities of Aggregated Lognormal Random Fields with Small Noise

Xiaoou Li, Jingchen Liu
Department of Statistics, Columbia University, 1255 Amsterdam Avenue, New York, New York, 10027, USA, xl2306@columbia.edu, jcliu@stat.columbia.edu

Gongjun Xu
School of Statistics, University of Minnesota, 313 Ford Hall, 224 Church St. SE Minneapolis, Minnesota 55455, USA, xuxxx360@umn.edu

We develop asymptotic approximations for the tail probabilities of integrals of lognormal random fields taking the form $\int_T e^{\sigma f(t)+\mu(t)} m(dt)$ where f is a Gaussian random field. We consider the asymptotic regime that the variance of the random field σ^2 converges to zero. Under this setting, the integral converges to its limiting value $\int_T e^{\mu(t)} m(dt)$. The tail probabilities are evaluated at places that are $O(\sigma^\alpha)$ distance away from this limiting value for some $\alpha \in (0,1)$. This analysis is of interest on considering short term portfolio risk analysis (such as daily performance), for which the variances of log-returns could be as small as a few percent.

Key words: Gaussian process, exponential integral, change of measure
MSC2000 subject classification: 60G15; 60G60
OR/MS subject classification: Primary: Decision analysis; secondary: Risk
History: Received June 28 2014; revised December 4 2014; accepted February 2 2015.

---

1. Introduction Let $\{f(t): t \in T\}$ be a zero-mean continuous Gaussian random field living on a compact set $T \subset R^d$. For a continuous and deterministic function $\mu(t)$ and a finite positive measure $m(\cdot)$ on T, we are interested in the probability

$$v(\sigma) = P\Big(\int_T e^{\sigma f(t)+\mu(t)} m(dt) > b\Big), \quad \text{as } \sigma \to 0, \tag{1}$$

where

$$b = \int_T e^{\mu(t)} m(dt) + \kappa \sigma^\alpha \tag{2}$$

for some constants $\kappa > 0$ and $0 < \alpha < 1$. We consider two cases: m is a discrete measure with finitely many point masses and m is the Lebesgue measure.

Motivation. The integral of lognormal random fields is the central quantity of many probabilistic models in portfolio risk analysis, spatial point processes, etc. (see, e.g., Liu and Xu [12, 14]). The current analysis is of interest particularly for risk analysis of short-term behavior of a large size portfolio under high correlations. We elaborate more on this application. Consider a portfolio consisting of n assets denoted by S_1, ..., S_n, each of which is associated to a weight, denoted by w_1, ..., w_n. The total value is $S = \sum_{i=1}^n w_i S_i$. Of interest is the tail behavior of S. A stylized model assumes that S_i's are lognormal random variables. Then, the total value is the sum of n correlated lognormal random variables (Ahsan [1], Duffie and Pan [6], Glasserman et al. [10], Basak and Shapiro [3], Deutsch [5], Foss and Richards [8]). Under such a setting, one may employ a latent space approach by embedding S_1, ..., S_n in a Gaussian process. More precisely, we construct a

Gaussian process $f(t)$ and a deterministic function $w(t)$. For each $1 \le i \le n$ there exists $t_i \in T$ such that $S_i = e^{f(t_i)}$ and $w_i = w(t_i)$. An interesting situation is that the portfolio size is large and the asset prices become highly correlated. Then the set $\{t_1, ..., t_n\}$ becomes dense in T. Ultimately, as the portfolio size tends to infinity, the limiting value of the unit share price becomes

$$\frac{1}{n}\sum_{i=1}^{n} w(t_i)S_i \to \int_T w(t)e^{f(t)}m(dt)$$

where $m(\cdot)$ is the limiting distribution of $\{t_1, ..., t_n\}$.

Upon considering the short-term behavior of the portfolio, the variance of each asset S_i is usually small. For instance, the variance of the daily log-return of a liquid stock is usually on the order of a few percent that corresponds to the variance of f. Thus, we introduce an additional overall volatility parameter σ and consider

$$\int_T w(t)e^{\sigma f(t)}m(dt).$$

Sending σ to zero is equivalently to considering a very short-term return of the portfolio. We are interested in that $\int_T w(t)e^{\sigma f(t)}m(dt)$ deviates from its limiting value, $\int_T w(t)m(dt)$, by an amount $\kappa\sigma^\alpha$ that is slightly larger than σ, i.e., the target probability in (1) with $e^{\mu(t)} = w(t)$. For instance, if σ is on the order of a few percent, then $\kappa\sigma^\alpha$ is of a larger order such as ten percent. In order to have the probability $v(\sigma)$ eventually converging to zero, it is necessary to keep α strictly less than one.

Related works. The tail probabilities of integrals of lognormal fields have been studied both intensively and extensively in the literature, most of which focuses on the asymptotic regime that b tends to infinity and σ is fixed. Asmussen and Rojas-Nandayapa [2] and Gao et al. [9] study tail probabilities and the density functions for summations of lognormal random variables. The distributions of integrals of geometric Brownian motions are studied in Yor [16] and Dufresne [7]. For more general continuous Gaussian random fields, Liu [11] and Liu and Xu [12] derive the asymptotic approximations of $P(\int_T e^{f(t)}dt > b)$ as $b \to \infty$ when $f(t)$ is a three-time differentiable Gaussian random field. Under similar conditions, Liu and Xu [14] characterize the conditional probabilities $P(\ \cdot \mid \int_T e^{\sigma f(t)+\mu(t)}dt > b)$ as $b \to \infty$ and efficient Monte Carlo estimators of $v(\sigma)$ are then constructed. The corresponding density function is studied in Liu and Xu [13].

This paper considers the asymptotic regime that σ tends to zero. We develop asymptotic approximations of the tail probabilities under very weak regularity conditions. The tail behaviors under small noise are different from the cases when b tends to infinity and σ is fixed. For the latter case the most likely sample paths typically admit the so-called *one-big-jump principle*, that is, the high value of the exponential integral is due to the high excursion of $f(t)$ at one location and the integral in a small region around the maximum of $f(t)$ is dominating. For case that σ converges to zero, there is not a small dominating region and the integral on every piece of the region has a contribution. This feature is often observed in the portfolio risk analysis. Suppose that a large portfolio has a 10% downturn in one day. It is very likely to observe that most stocks in the portfolio has a substantial negative return lead by a few (or sector of) names whose returns are the most negative among all.

In addition to the right tail, with completely analogous analysis, we provide approximations of the left tail probabilities

$$v_l(\sigma) = P\Big(\int_T e^{\sigma f(t)+\mu(t)}m(dt) < b\Big), \quad \text{for } b = \int_T e^{\mu(t)}m(dt) - \kappa\sigma^\alpha. \tag{3}$$

The rest of the paper is organized as follows. The main approximation results are presented in Section 2. Section 3 includes the proofs of the theorems presented in Section 2.

2. Main results

2.1. Asymptotic approximations We start the discussion with the case when $m(\cdot)$ is the Lebesgue measure. Let

$$C(s,t) = E(f(s)f(t))$$

be the covariance function of the Gaussian random field $f(t)$ and assume that $C(s,t)$ is positive definite. Let $\mathcal{C}(T)$ denote the set of continuous functions on T. Define a map $K : \mathcal{C}(T) \mapsto [0,\infty]$ as follows: for each $x(\cdot) \in \mathcal{C}(T)$,

$$K(x) = \int_T \int_T x(s)C(s,t)x(t)dsdt \tag{4}$$

that is the squared Mahalanobis distance induced by C. Define a linear map $\mathbf{C} : \mathcal{C}(T) \mapsto \mathcal{C}(T)$

$$\mathbf{C}(x)(t) = \int_T C(s,t)x(s)ds.$$

We consider the optimization problem

$$K_\sigma^* = \min_{x\in\mathcal{C}(T)} K(x) \quad \text{subject to the constraints} \quad \int_T e^{\sigma \mathbf{C}(x)(t)+\mu(t)}dt \geq b \text{ and } \sup_{t\in T}|x(t)| \leq \sigma^{\alpha-1-\varepsilon}, \tag{5}$$

for some $\varepsilon \in (0, \min(\alpha, 1-\alpha))$. For σ sufficiently small, the above optimization problem has a unique solution and it does not depend on the choice of ε. The properties of the solution will be discussed later in this section. Now we present the first result.

THEOREM 1. *For $0 < \alpha < 1$, suppose that the covariance function $C(s,t)$ is positive definite and m is the Lebesgue measure. Let K_σ^* be defined as in (5). We have the following approximation of $v(\sigma)$*

$$v(\sigma) = (c_1 + o(1))\sigma^{1-\alpha} \exp\Big(-\frac{1}{2}K_\sigma^*\Big), \quad \text{as } \sigma \to 0, \tag{6}$$

where

$$c_1 = \kappa^{-1}\{(2\pi)^{-1}K(e^{\mu(\cdot)})\}^{1/2} \tag{7}$$

and the constant κ appears initially in (2).

The above theorem provides an almost explicit approximation of $v(\sigma)$. The implicitly part lies in K_σ^* that is unfortunately not in a closed form. We will later present an iterative algorithm to compute K_σ^* numerically. To maintain the approximation accuracy in Theorem 1, we need to have the computational error reduced to the level of $o(1)$. Due to the technical complication and also to smooth the discussion, we delay this topic to the following subsection. In the meantime, we provide the first order approximation of K_σ^* in the following proposition. This approximation is sufficient to provide an exponential decay rate of $v(\sigma)$.

PROPOSITION 1. *Under the conditions of Theorem 1, for σ sufficiently small, we have the following results.*

(i) For $0 < \alpha < 1$, the optimization problem (5) has a unique solution, denoted by $x^*(t)$.

(ii) We have the following approximations as $\sigma \to 0$

$$\begin{aligned} x^*(t) &= (1+o(1))\kappa\sigma^{\alpha-1}\frac{e^{\mu(t)}}{\int_{T\times T} C(s,t)e^{\mu(s)+\mu(t)}dsdt}, \\ K_\sigma^* &= (1+o(1))\kappa^2\sigma^{2\alpha-2}K(e^{\mu(\cdot)})^{-1}. \end{aligned} \tag{8}$$

The first $o(1)$ term is uniform in $t \in T$ as $\sigma \to 0$.

The approximations in Proposition 1(ii) are obtained via the first order expansion of the integral $\int_T e^{\sigma f(t)+\mu(t)}dt$. Better approximations of x^* and K_σ^* can be obtained by expanding higher orders. As mentioned previously, to maintain an accurate approximation, we need to reduce the accuracy to the level $o(1)$. The necessary order of expansions in fact depends on α and the derivation is doable but very tedious. Thus, we seek for alternative numerical methods presented in the sequel. Combining Theorem 1 and Proposition 1 we have the following approximation of $\log v(\sigma)$.

COROLLARY 1. *Under the conditions of Theorem 1, for $0<\alpha<1$, as $\sigma\to 0$,*

$$\log v(\sigma) = -(1+o(1))\frac{1}{2}\kappa^2\sigma^{2\alpha-2}K(e^{\mu(\cdot)})^{-1}.$$

REMARK 1. An intuitive understanding of the above approximation result is given as follows. As $\sigma\to 0$, we approximate the interval by Taylor expansion $\int_T e^{\sigma f(t)+\mu(t)}dt \approx \int_T e^{\mu(t)}(1+\sigma f(t))dt$. This suggests that $v(\sigma)\approx P(\int_T e^{\mu(t)}f(t)dt > \kappa\sigma^{\alpha-1})$. Since $\int_T e^{\mu(t)}f(t)dt$ is a Gaussian random variable with zero mean and finite variance, we have approximation $v(\sigma)\approx \exp\{-O(\kappa^2\sigma^{2\alpha-2})\}$. This gives the order of the leading term in Theorem 1.

We now consider that $m(\cdot)$ is a discrete measure on T with finitely many point masses. For simplicity, we write the random field in terms of a random vector $X=(X_1,..,X_n)^T$ that has a positive definite covariance matrix Σ. Furthermore, we replace the function $\mu(t)$ with a vector $\mu=(\mu_1,..,\mu_n)^T$. The probability $v(\sigma)$ becomes

$$v(\sigma) = P\Big(\sum_{i=1}^n e^{\sigma X_i+\mu_i} > b\Big). \tag{9}$$

Similarly to the continuous case, we define the squared Mahalanobis distance for $x\in R^n$,

$$\tilde K(x) = x^T\Sigma x.$$

We further define $\tilde K_\sigma^*$ through the optimization problem

$$\tilde K_\sigma^* = \min_x \tilde K(x) \quad \text{subject to the constraint} \quad \sum_{i=1}^n e^{\sigma(\Sigma x)_i+\mu_i} \ge b, \tag{10}$$

where $(\Sigma x)_i$ is the ith element of Σx. The next theorem presents an approximation of $v(\sigma)$ for $0<\alpha<1$, which is the discrete analogue of Theorem 1.

THEOREM 2. *The covariance matrix Σ is positive definite. Let $\tilde K_\sigma^*$ be defined as in (10) and b be defined as in (2). For $0<\alpha<1$, we have*

$$v(\sigma) = (c_2+o(1))\sigma^{1-\alpha}\exp\Big(-\frac{\tilde K_\sigma^*}{2}\Big), \quad \text{as } \sigma\to 0, \tag{11}$$

where $c_2=\kappa^{-1}\sqrt{(2\pi)^{-1}y^{*T}\Sigma y^*}$ and

$$y^* = (e^{\mu_1},...,e^{\mu_n})^T. \tag{12}$$

We have the following discrete analogue of Proposition 1.

PROPOSITION 2. *Under the conditions of Theorem 2, for $0<\alpha<1$, we have the following results.*

(i) The optimization problem (10) has a unique solution $x^*\in R^n$.

(ii) We have the following approximation

$$\begin{aligned} x^* &= (1+o(1))\kappa\sigma^{\alpha-1}(y^{*T}\Sigma y^*)^{-1}y^*, \\ \tilde{K}^*_\sigma &= (1+o(1))\kappa^2\sigma^{2\alpha-2}(y^{*T}\Sigma y^*)^{-1}, \end{aligned}$$

where y^* is given as in (12).

Combining the above proposition and Theorem 2, we have the following approximation of $\log v(\sigma)$.

Corollary 2. *Under the conditions of Theorem 2, for $0<\alpha<1$, we have as $\sigma\to 0$*

$$\log(v(\sigma)) = -(1+o(1))\frac{1}{2}\kappa^2(y^{*T}\Sigma y^*)^{-1}\sigma^{2\alpha-2}.$$

The approximations of the left-tail probabilities can be derived similarly as those of the right tail. Therefore, we present the results as corollaries and omit the proof. For the case when $m(\cdot)$ is the Lebesgue measure, we redefine K^*_σ through the optimization problem

$$\begin{aligned} K^*_\sigma = \min_{x\in\mathcal{C}(T)} K(x) \quad &\text{subject to the constraints} \\ &\int_T e^{\sigma\mathbf{C}(x)(t)+\mu(t)}dt \le \int_T e^{\mu(t)}dt - \kappa\sigma^\alpha \text{ and } \sup_{t\in T}|x(t)|\le\sigma^{\alpha-1-\varepsilon}. \end{aligned} \tag{13}$$

Corollary 3. *With K^*_σ defined in (13), we have*

$$P\Big(\int_T e^{\sigma f(t)+\mu(t)}dt < \int_T e^{\mu(t)}dt - \kappa\sigma^\alpha\Big) = (c_1+o(1))\sigma^{1-\alpha}\exp\Big(-\frac{1}{2}K^*_\sigma\Big), \quad \text{as } \sigma\to 0,$$

where c_1 is given as in (7).

When $m(\cdot)$ is a discrete measure with finitely many point masses, we redefine the optimization problem as

$$\tilde{K}^*_\sigma = \min_x \tilde{K}(x) \quad \text{subject to} \quad \sum_{i=1}^n e^{\sigma(\Sigma x)_i+\mu_i} \le \sum_{i=1}^n e^{\mu_i} - \kappa\sigma^\sigma. \tag{14}$$

Corollary 4. *With $\tilde{K}^*_\sigma$ defined in (14), we have*

$$P\Big(\sum_{i=1}^n e^{\sigma X_i+\mu_i} < \sum_{i=1}^n e^{\mu_i} - \kappa\sigma^\alpha\Big) = (c_2+o(1))\sigma^{1-\alpha}\exp\Big(-\frac{\tilde{K}^*_\sigma}{2}\Big), \quad \text{as } \sigma\to 0,$$

where $c_2=\kappa^{-1}\sqrt{(2\pi)^{-1}y^{*T}\Sigma y^*}$.

2.2. Numerical approximation for K^*_σ As discussed previously, K^*_σ is not a closed form expression. In this section, we present an iterative algorithm to solve (5) and m is the Lebesgue measure. The case of discrete measure is similar and therefore is omitted. Let

$$\mathcal{B} = \{x\in\mathcal{C}(T): \|x\|_\infty \le \sigma^{\alpha-1-\varepsilon}\},$$

where $\|x\|_\infty = \sup_{t\in T}|x(t)|$. Define the function $\Lambda(\cdot):\mathcal{B}\to[0,+\infty)$ such that, for each $x\in\mathcal{B}$, $\lambda = \Lambda(x)$ solves the following equation

$$\int_T \exp\Big\{\sigma\lambda\mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu})(t)+\mu(t)\Big\}dt = b. \tag{15}$$

The next proposition ensures that $\Lambda(\cdot)$ is well defined.

PROPOSITION 3. *For each $x\in\mathcal{B}$, there is a unique solution $\Lambda(x)$ satisfying equation (15). Moreover, $0\le\Lambda(x)\le\kappa_c\sigma^{\alpha-1}$, where κ_c is a positive constant depending only on the covariance function C and the mean function μ.*

We further define the operator $\mathbf{S}:\mathcal{B}\to\mathcal{B}$ by

$$\mathbf{S}(x)(t)=\Lambda(x)e^{\sigma\mathbf{C}(x)(t)+\mu(t)}. \tag{16}$$

Our algorithm to compute K_σ^* is based on the following proposition.

PROPOSITION 4. *$\mathbf{S}$ is a contraction mapping over $\mathcal{B}$, that is, for $x,y\in\mathcal{B}$,*

$$\|\mathbf{S}(x)-\mathbf{S}(y)\|_\infty\le\kappa_0\sigma^\alpha\|x-y\|_\infty, \tag{17}$$

where κ_0 is a positive constant depending only on the covariance function C and the mean function μ. Furthermore, the solution $x^*(\cdot)$ to the optimization problem (5) is the unique fixed point of $\mathbf{S}$, that is, $x^*=\mathbf{S}(x^*)$.

With the above proposition, we present an iterative algorithm to compute x^* using the above contraction mapping theorem.

1. Let
$$\hat{x}_0^*=\kappa\sigma^{\alpha-1}\frac{e^{\mu(t)}}{\int_T\int_T C(s,t)e^{\mu(s)+\mu(t)}dsdt}.$$
2. For each k, compute $\hat{x}_k^*$ according to
$$\hat{x}_k^*=\mathbf{S}(\hat{x}_{k-1}^*).$$

We iterate step 2 until convergence. According to the contraction mapping theorem, the rate of convergence is

$$\|\hat{x}_k^*-x^*\|_\infty\le(\kappa_0\sigma^\alpha)^k\|\hat{x}_0^*-x^*\|_\infty=O(\sigma^{\alpha k+\alpha-1}).$$

If we run the algorithm for $k>2(1-\alpha)/\alpha$ iterations, then $\|\hat{x}_k^*-x^*\|_\infty=O(\sigma^{\alpha k+\alpha-1})=o(\sigma^{1-\alpha})$. We obtain that $|K(\hat{x}_k^*)-K_\sigma^*|=o(\sigma^{1-\alpha})$ and the asymptotic results in the previous theorems still hold by replacing K_σ^* with $K(\hat{x}_k^*)$.

3. Proof In this section, we present the proofs of Theorem 1 and Propositions 1, 3, and 4. The proofs for Theorem 2 and Proposition 2 are completely analogous to those of Theorem 1 and Proposition 1 and therefore are omitted.

We begin with some useful lemmas. The following lemma is known as the Borell-TIS lemma, which is proved independently by Borell [4] and Tsirelson et al. [15].

LEMMA 1 **(Borell-TIS)**. *Let $f(t)$, $t\in\mathcal{U}$, $\mathcal{U}$ is a parameter set, be a mean zero Gaussian random field. f is almost surely bounded on $\mathcal{U}$. Then, $E[\sup_\mathcal{U} f(t)]<\infty$, and*

$$P\left(\sup_{t\in\mathcal{U}}f(t)-E[\sup_{t\in\mathcal{U}}f(t)]\ge b\right)\le\exp\left(-\frac{b^2}{2\sigma_\mathcal{U}^2}\right),$$

where $\sigma_\mathcal{U}^2=\sup_{t\in\mathcal{U}}Var[f(t)]$.

The Borell-TIS lemma provides a general bound of the tail probabilities of $\sup_t f(t)$. In most cases, $E[\sup_t f(t)]$ is much smaller than b. Thus, for b that is sufficiently large, the tail probability can be further bounded by:

$$P\left(\sup_{t\in T}f(t)>b\right)\le\exp\left(-\frac{b^2}{4\sigma_T^2}\right). \tag{18}$$

To prove Theorem 1, the following lemma shows that $f(t)$ can be localized to the event

$$\mathcal{L}=\Big\{f(t):\sup_{t\in T}|f(t)|\le\kappa_f\sigma^{\alpha-1}\Big\},$$

and we only need to focus on $\mathcal{L}$ for our analysis.

LEMMA 2. *There exists a positive constant κ_f sufficiently large such that*

$$P\Big(\sup_{t\in T}|f(t)|>\kappa_f\sigma^{\alpha-1}\Big)=o(1)\sigma^{1-\alpha}\exp\Big(-\frac{1}{2}K_\sigma^*\Big).$$

Proof of Lemma 2. According to Proposition 1, whose proof is independent of the current one, $K_\sigma^*=(1+o(1))\kappa^2\sigma^{2\alpha-2}K(e^{\mu(\cdot)})^{-1}$. We choose the constant $\kappa_f>2\sigma_T\kappa\sqrt{K(e^{\mu(\cdot)})^{-1}}$, then inequality (18) implies that

$$P\Big(\sup_{t\in T}|f(t)|>\kappa_f\sigma^{\alpha-1}\Big)\le 2\exp\left(-\kappa^2\sigma^{2\alpha-2}K(e^{\mu(\cdot)})^{-1}\right)=o(1)\sigma^{1-\alpha}\exp\Big(-\frac{1}{2}K_\sigma^*\Big),$$

which yields the desired result. ∎

We proceed to the proof of Theorem 1. We use a change of measure technique to derive the asymptotic approximation. The change of measure is constructed such that it focuses on the most likely sample path corresponding to the solution to the optimization problem (5). The theoretical properties of the optimization problem (5) are established in Propositions 1, 3 and 4. These three propositions are the key elements of the proof.

Proof of Theorem 1. Let $x^*(t)$ be the solution to (5). We define the exponential change of measure

$$\frac{dQ}{dP}=\exp\Big(\int_T x^*(t)f(t)dt-\frac{1}{2}\int_T\int_T x^*(s)C(s,t)x^*(t)dsdt\Big).$$

The introduced change of measure Q defines a translation of the original Gaussian random field $f(t)$. We state this result in the next lemma, whose proof is delayed after the proof of Theorem 1.

LEMMA 3. *Under measure Q, $f(t)$ is a Gaussian random field with mean function $\mathbf{C}(x^*)(t)$ and covariance function $C(s,t)$.*

According to Lemma 2,

$$P\left(\int_T e^{\sigma f(t)+\mu(t)}>b,\mathcal{L}^c\right)=o(1)\sigma^{1-\alpha}\exp\Big(-\frac{1}{2}K_\sigma^*\Big).$$

Therefore, we only need to consider $P(\int_T e^{\sigma f(t)+\mu(t)}>b,\mathcal{L})$. By means of the change of measure Q, we have

$$\begin{aligned}&P\left(\int_T e^{\sigma f(t)+\mu(t)}>b,\mathcal{L}\right)\\&=E^Q\left[\frac{dP}{dQ};\ \int_T e^{\sigma f(t)+\mu(t)}>b,\ \mathcal{L}\right]\\&=\exp\left(\frac{1}{2}\int_{T\times T}x^*(s)C(s,t)x^*(t)dsdt\right)E^Q\left[e^{-\int_T x^*(t)f(t)dt};\int_T e^{\sigma f(t)+\mu(t)}dt>b,\mathcal{L}\right],\end{aligned}\qquad(19)$$

where E^Q denotes the expectation with respect to the measure Q. Let

$$f^*=\mathbf{C}(x^*).$$

With this notation, we have

$$\int_T e^{\sigma f^*(t)+\mu(t)}dt = b, \quad \int_T f^*(t)x^*(t)dt = \int_{T\times T} x^*(s)C(s,t)x^*(t)dsdt.$$

The random field $f^*(t)+f(t)$ under P has the same distribution as $f(t)$ under Q. Thus, we replace the probability measure Q and f with P and f^*+f in (19) and obtain

$$\begin{aligned} &P\left(\int_T e^{\sigma f(t)+\mu(t)} > b, \mathcal{L}\right) \\ &= \exp\left(\frac{1}{2}\int_{T\times T} x^*(s)C(s,t)x^*(t)dsdt\right) E\left[e^{-\int_T x^*(t)(f^*(t)+f(t))dt}; \int_T e^{\sigma(f^*(t)+f(t))+\mu(t)}dt > b, \mathcal{L}\right] \\ &= \exp\left(-\frac{1}{2}\int_{T\times T} x^*(s)C(s,t)x^*(t)dsdt\right) E\left[e^{-\int_T x^*(t)f(t)dt}; \int_T (e^{\sigma f(t)}-1)w(dt) > 0, \mathcal{L}\right], \end{aligned} \tag{20}$$

where

$$w(dt) = \frac{y^*(t)dt}{\int_T y^*(s)ds} \text{ and } y^*(t) = e^{\sigma f^*(t)+\mu(t)}.$$

We define

$$F = \left\{\int_T (e^{\sigma f(t)}-1)w(dt) > 0\right\}.$$

By the fact that $e^x - 1 \geq x$, we have

$$\int_T (e^{\sigma f(t)}-1)w(dt) \geq \int_T \sigma f(t)w(dt).$$

Thus, F can be written as the union of two disjoint sets, $F = F_1 \cup F_2$, where

$$F_1 = \left\{\int_T f(t)w(dt) > 0\right\} \text{ and } F_2 = \left\{\int_T f(t)w(dt) < 0, \int_T (e^{\sigma f(t)}-1)w(dt) > 0\right\}.$$

Thus, the expectation in (20) can be written as

$$E\left[e^{-\int_T x^*(t)f(t)dt}; \int_T (e^{\sigma f(t)}-1)w(dt) > 0, \mathcal{L}\right] = E\left[e^{-\int_T x^*(t)f(t)dt}; F_1, \mathcal{L}\right] + E\left[e^{-\int_T x^*(t)f(t)dt}; F_2, \mathcal{L}\right]. \tag{21}$$

We calculate each of the two terms on the right-hand side of the above equation separately. First, we compute

$$E\left[e^{-\int_T x^*(t)f(t)dt}; \int_T f(t)w(dt) > 0, \mathcal{L}\right]. \tag{22}$$

According to Proposition 4, whose proof is independent of the current one, x^* is the fixed point of the contraction map $\mathbf{S}$ and thus

$$x^*(t) = \mathbf{S}(x^*)(t) = \Lambda(x^*)e^{\sigma \mathbf{C}(x^*)(t)+\mu(t)} = \Lambda(x^*)y^*(t).$$

Therefore, $x^*(t)$ and $y^*(t)$ are different by a factor $\Lambda(x^*)$. Thus, $\int_T x^*(t)f(t)dt$ and $\int_T f(t)w(dt)$ are different by a factor $\int_T x^*(t)dt$. Thanks to Proposition 1(ii), we have

$$\int_T x^*(t)dt = (1+o(1))\frac{\kappa\sigma^{\alpha-1}\int_T e^{\mu(t)}dt}{\int_{T\times T} C(s,t)e^{\mu(s)+\mu(t)}dsdt}.$$

As the result, we have

$$\int_T x^*(t)f(t)dt = \int_T x^*(t)dt \int_T f(t)w(dt)$$
$$= (1+o(1))\frac{\kappa\sigma^{\alpha-1}\int_T e^{\mu(t)}dt}{\int_{T\times T} C(s,t)e^{\mu(s)+\mu(t)}dsdt}\int_T f(t)w(dt). \tag{23}$$

Define

$$\Delta = \frac{\kappa\sigma^{\alpha-1}\int_T e^{\mu(t)}dt}{\int_{T\times T} C(s,t)e^{\mu(s)+\mu(t)}dsdt}.$$

The expectation (22) can be computed as follows

$$\begin{aligned}
&E\left[e^{-\int_T x^*(t)f(t)dt}; \int_T f(t)w(dt) > 0, \mathcal{L}\right] \\
&= E\left[e^{-(1+o(1))\Delta\int_T f(t)w(dt)}; \int_T f(t)w(dt) > 0, \mathcal{L}\right] \\
&= (1+o(1))E\left[e^{-(1+o(1))\Delta\int_T f(t)w(dt)}; \int_T f(t)w(dt) > 0\right] \\
&= (1+o(1))\frac{1}{\Delta\sqrt{2\pi Var(\int_T f(t)w(dt))}}.
\end{aligned}$$

The second step in the above derivation is due to the fact that $P(\mathcal{L}) \to 1$ for κ_f chosen sufficiently large. Furthermore, notice that $w(t) = (1+o(1))e^{\mu(t)}/\int e^{\mu(s)}ds$. Then,

$$Var\left(\int_T f(t)w(dt)\right) = (1+o(1))\frac{\int_{T\times T} e^{\mu(s)+\mu(t)}C(s,t)dsdt}{(\int_T e^{\mu(t)}dt)^2}$$

and

$$E\left[e^{-\int_T x^*(t)f(t)dt}; \int_T f(t)w(dt) > 0, \mathcal{L}\right] = (1+o(1))\kappa^{-1}\sigma^{1-\alpha}\sqrt{(2\pi)^{-1}\int_{T\times T} C(s,t)e^{\mu(s)+\mu(t)}dsdt}.$$

Thus, we conclude the derivation of the first expectation on the right-hand side of (21).

Now we proceed to the second expectation term. On the set $\mathcal{L}$, by Taylor's expansion, we have that $e^{\sigma f(t)} - 1 \le \sigma f(t) + \sigma^2 f^2(t)$ and thus

$$\int_T (e^{\sigma f(t)} - 1)w(dt) \le \int_T \sigma f(t)w(dt) + \int_T \sigma^2 f^2(t)w(dt).$$

So the event $\{\int_T (e^{\sigma f(t)} - 1)w(dt) \ge 0\}$ is a subset of $\{\int_T [f(t) + \sigma f^2(t)]w(dt) \ge 0\}$. This gives an upper bound of the expectation

$$E\left[e^{-\int_T x^*(t)f(t)dt}; F_2, \mathcal{L}\right] \le E\left[e^{-\int_T x^*(t)f(t)dt}; \int_T [f(t) + \sigma f^2(t)]w(dt) \ge 0, \int_T f(t)w(dt) < 0, \mathcal{L}\right].$$

We write

$$Z_1 = -\int_T f(t)w(dt) \text{ and } Z_2 = \int_T f^2(t)w(dt).$$

From (23), the right-hand side of the above inequality can be written as

$$E[e^{\Delta Z_1}; Z_1 > 0, Z_2 \ge Z_1/\sigma, \mathcal{L}].$$

On the set $\{0<Z_1\le\sigma^{1-\alpha+\varepsilon}\}$, this expectation is negligible as $\Delta=O(\sigma^{\alpha-1})$, that is,

$$E[e^{\Delta Z_1};0<Z_1<\sigma^{1-\alpha+\varepsilon}]=O(P(0<Z_1<\sigma^{1-\alpha+\varepsilon}))=o(1). \tag{24}$$

Furthermore, on the set $\mathcal{L}$, we have $\sup_t|f(t)|\le\kappa_f\sigma^{\alpha-1}$ and thus $Z_1<\sigma^{\alpha-1-\varepsilon}$ for ε and σ sufficiently small. Therefore, we only need to focus on the expectation

$$E\left[e^{\Delta Z_1};\sigma^{1-\alpha+\varepsilon}<Z_1<\sigma^{\alpha-1-\varepsilon},Z_2>Z_1/\sigma\right]=\int_{\sigma^{1-\alpha+\varepsilon}}^{\sigma^{\alpha-1-\varepsilon}}e^{\Delta z}P(Z_2>z/\sigma|Z_1=z)p_{Z_1}(z)dz, \tag{25}$$

where $p_{Z_1}(z)$ is the density function of Z_1. We need the following lemma.

Lemma 4. *For $z\in[\sigma^{1-\alpha+\varepsilon},\sigma^{\alpha-1-\varepsilon}]$, there exists a constant $\varepsilon_0>0$ such that*

$$P(Z_2>z/\sigma|Z_1=z)\le e^{-\varepsilon_0 z/\sigma}. \tag{26}$$

Lemma 4 implies that the expectation (25) is bounded by

$$(25)\le\int_{\sigma^{1-\alpha+\varepsilon}}^{\sigma^{\alpha-1-\varepsilon}}e^{-(\varepsilon_0/\sigma-\Delta)z}p_{Z_1}(z)dz=\int_{\sigma^{1-\alpha+\varepsilon}}^{\sigma^{\alpha-1-\varepsilon}}e^{-(1+o(1))\varepsilon_0 z/\sigma}p_{Z_1}(z)dz=O(\sigma). \tag{27}$$

Combining the results in (24) and (27), we have $E[e^{-\int_T x^*(t)f(t)dt};F_2,\mathcal{L}]=o(1)$ and Theorem 1 is proved. ■

Proof of Lemma 3. It is sufficient to show that, for any finite subset $\{t_1,\dots,t_k\}\in T$, the moment generating function of $(f(t_1),\dots,f(t_k))$ under the measure Q is the same as that of the multivariate normal distribution with mean $(\mathbf{C}(x^*)(t_1),\dots,\mathbf{C}(x^*)(t_k))$ and covariance matrix $\{C(t_i,t_j)\}_{i,j=1,\dots,k}$. For any $(\lambda_1,...,\lambda_k)\in\mathbb{R}^k$, we have

$$\begin{aligned}
&E^Q\Big[\exp\{\lambda_1 f(t_1)+\cdots+\lambda_k f(t_k)\}\Big]\\
&=E\left[\frac{dQ}{dP}\exp\{\lambda_1 f(t_1)+\cdots+\lambda_k f(t_k)\}\right]\\
&=E\left[\exp\left\{\int_T x^*(t)f(t)dt-\frac{1}{2}\int_T\int_T x^*(s)C(s,t)x^*(t)dsdt+\lambda_1 f(t_1)+\cdots+\lambda_k f(t_k)\right\}\right]\\
&=\exp\left\{-\frac{1}{2}\int_T\int_T x^*(s)C(s,t)x^*(t)dsdt\right\}E\left[\exp\left\{\int_T x^*(t)f(t)dt+\lambda_1 f(t_1)+\cdots+\lambda_k f(t_k)\right\}\right]\\
&=\exp\left\{-\frac{1}{2}\int_T\int_T x^*(s)C(s,t)x^*(t)dsdt+\frac{1}{2}Var\left(\int_T x^*(t)f(t)dt+\lambda_1 f(t_1)+\cdots+\lambda_k f(t_k)\right)\right\}\\
&=\exp\left\{\sum_{i=1}^k\lambda_i\mathbf{C}(x^*)(t_i)+\frac{1}{2}\sum_i^k\sum_{j=1}^k\lambda_i\lambda_j C(t_i,t_j)\right\},
\end{aligned}$$

which is the moment generating function of the target multivariate normal distribution. This completes the proof. ■

Proof of Lemma 4. Conditional on $Z_1=z$, $\{f(t):t\in T\}$ is still a Gaussian random field, with the mean and variance given as follows:

$$\tilde{\mu}(t)=E(f(t)|Z_1=z)=-\frac{\int_T C(s,t)w(ds)}{\int_{T\times T}C(s,t)w(ds)w(dt)}\cdot z, \tag{28}$$
$$Var(f(t)|Z_1=z)=C(t,t)-\left(\int_{T\times T}C(s,t)w(ds)w(dt)\right)^{-1}\left(\int_T C(s,t)w(ds)\right)^2.$$

We write the conditional random field as $f(t) = \tilde{\mu}(t) + g(t)$, then the probability in (26) is bounded by

$$P\left(\int_T \{\tilde{\mu}(t) + g(t)\}^2 w(dt) > z/\sigma\right) \le P\left(\sup_{t\in T} |\tilde{\mu}(t)| + \sup_T |g(t)| > \sqrt{z/\sigma}\right).$$

According to (28), for $z \in [\sigma^{1-\alpha+\varepsilon}, \sigma^{\alpha-1-\varepsilon}]$, we have $\sup_{t\in T} |\tilde{\mu}(t)| = O(z) = o(1)\sqrt{z/\sigma}$. So the above probability can be further bounded by

$$P\left(\sup_T |g(t)| > (1 + o(1))\sqrt{z/\sigma}\right).$$

We obtain (26) by applying Lemma 1. This concludes our proof. ■

The proof of Proposition 1 needs the results of Propositions 3 and 4. Thus, we present the proofs of these two propositions first.

Proof of Proposition 3. For $x \in \mathcal{B}$, we define

$$h(\lambda) = \int_T \exp\left(\sigma\lambda\mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu})(t) + \mu(t)\right) dt.$$

We have

$$\begin{aligned} h(\lambda) &\ge \int_T e^{\mu(t)}(1 + \sigma\lambda\mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu})(t))dt \\ &= \int_T e^{\mu(t)}dt + \sigma\lambda \int_T e^{\mu(t)}\mathbf{C}(e^{\mu}(1 + o(1)))(t)dt \\ &= \int_T e^{\mu(t)}dt + (1 + o(1))\sigma\lambda \int_{T\times T} e^{\mu(s)}C(s,t)e^{\mu(t)}dsdt. \end{aligned} \tag{29}$$

The second equality holds because $\sigma\mathbf{C}(x) = O(\sigma^{\alpha-\varepsilon}) = o(1)$. If $h(\lambda) = b$, then, together with the fact that $b = \int_T e^{\mu(t)}dt + \kappa\sigma^\alpha$, the above display suggests that

$$\lambda \le (1 + o(1))\kappa\sigma^{\alpha-1}(\int_T\int_T e^{\mu(s)}C(s,t)e^{\mu(t)}dsdt)^{-1}.$$

This means that the equation $h(\lambda) = b$ has no solution outside $[0, \kappa_c\sigma^{\alpha-1}]$ for some constant κ_c large.

For $\lambda \in [0, \kappa_c\sigma^{\alpha-1}]$, we obtain the following approximation by Taylor's expansion

$$h(\lambda) = \int_T e^{\mu(t)}dt + \sigma\lambda(1 + o(1))\int_T\int_T e^{\mu(s)}C(s,t)e^{\mu(t)}dsdt$$

and $h(\lambda)$ is approximately linear in λ as σ tends to 0. Because $h(0) < b$ and $h(\kappa_c\sigma^{\alpha-1}) > b$ for κ_c sufficiently large, there exists $\lambda \in [0, \kappa_c\sigma^{\alpha-1}]$ such that $h(\lambda) = b$. Moreover, for $\lambda \in [0, \kappa_c\sigma^{\alpha-1}]$,

$$h'(\lambda) = (1 + o(1))\sigma\int_T\int_T e^{\mu(s)}C(s,t)e^{\mu(t)}dsdt > 0,$$

so the solution is unique. ■

Proof for Proposition 4. We first show that $\mathbf{S}$ is a contraction mapping. According to the definition of $\mathbf{S}(x)$ in (16) we have that for $x, y \in \mathcal{B}$

$$\|\mathbf{S}(x) - \mathbf{S}(y)\|_\infty \le |\Lambda(x) - \Lambda(y)| \cdot \|e^{\sigma\mathbf{C}(x)+\mu}\|_\infty + \Lambda(y)\|e^{\sigma\mathbf{C}(x)+\mu} - e^{\sigma\mathbf{C}(y)+\mu}\|_\infty. \tag{30}$$

We give upper bounds for $|\Lambda(x)-\Lambda(y)|$ and $\|e^{\sigma\mathbf{C}(x)+\mu}-e^{\sigma\mathbf{C}(y)+\mu}\|_\infty$ separately. According to (15), we have

$$\int_T \exp\Big(\sigma\Lambda(x)\mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu})(t)+\mu(t)\Big)dt-\int_T \exp\Big(\sigma\Lambda(y)\mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu})(t)+\mu(t)\Big)dt \tag{31}$$
$$=\int_T \exp\Big(\sigma\Lambda(y)\mathbf{C}(e^{\sigma\mathbf{C}(y)+\mu})(t)+\mu(t)\Big)dt-\int_T \exp\Big(\sigma\Lambda(y)\mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu})(t)+\mu(t)\Big)dt. \tag{32}$$

We provide a bound for $|\Lambda(x)-\Lambda(y)|$ by deriving approximations for both sides of the above identity. Without loss of generality, we assume $\Lambda(x)>\Lambda(y)$. By exchanging the integration and derivative, the left-hand side is

$$(31)=\int_{\Lambda(y)}^{\Lambda(x)}\int_T \sigma\mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu})(t)\exp\Big(\sigma\lambda\mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu})(t)+\mu(t)\Big)dtd\lambda.$$

Thus, we have

$$(31)=(1+o(1))\sigma|\Lambda(x)-\Lambda(y)|\times\int_T \mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu})(t)e^{\mu(t)}dt.$$

Similarly, we have the right-hand side is

$$(32)\le(1+o(1))\sigma\Lambda(y)\int_T e^{\mu(t)}\mathbf{C}(e^{\sigma\mathbf{C}(x)+\mu}-e^{\sigma\mathbf{C}(y)+\mu})(t)dt.$$

Notice that $\|e^{\sigma\mathbf{C}(x)+\mu}-e^{\sigma\mathbf{C}(y)+\mu}\|_\infty\le O(\sigma)\|x-y\|_\infty$. Thus,

$$(32)=O(\sigma^2)\Lambda(y)\|x-y\|_\infty=O(\sigma^{\alpha+1})\|x-y\|_\infty.$$

By equating (31) and (32), we have

$$|\Lambda(x)-\Lambda(y)|=O(\sigma^\alpha)\|x-y\|_\infty. \tag{33}$$

Thus, the first term in (30) is bounded from the above by

$$|\Lambda(x)-\Lambda(y)|\cdot\|e^{\sigma\mathbf{C}(x)+\mu}\|_\infty=O(\sigma^\alpha)\|x-y\|_\infty.$$

We proceed to the second term on the right side of (30). By Taylor's expansion, we have

$$\|e^{\sigma\mathbf{C}(x)+\mu}-e^{\sigma\mathbf{C}(y)+\mu}\|_\infty\le O(\sigma)\|x-y\|_\infty. \tag{34}$$

Thus we obtain (17) by combining (30), (33), (34), and the fact that $\Lambda(x)\le\kappa_c\sigma^{\alpha-1}$.

We proceed to the proof that the fixed point of $\mathbf{S}$ is the solution to (5). We define set

$$\mathcal{M}=\left\{x\in\mathcal{C}(T):\int_T e^{\sigma\mathbf{C}(x)(t)+\mu(t)}dt\ge b \text{ and } \|x\|_\infty\le\sigma^{\alpha-1-\varepsilon}\right\}.$$

For $x\in\mathcal{M}$, define function $l(\eta)=\int_T e^{\sigma\eta\mathbf{C}(x)(t)+\mu(t)}dt$ that is monotonic increasing in η, so all solutions to the optimization problem (5) lie on the boundary set

$$\partial\mathcal{M}=\left\{x\in\mathcal{C}(T):\int_T e^{\sigma\mathbf{C}(x)(t)+\mu(t)}dt=b \text{ and } \|x\|_\infty\le\sigma^{\alpha-1-\varepsilon}\right\}.$$

We use arguments in calculus of variation to show the conclusion. Let g be an arbitrary continuous function on T and s be a scalar close to 0. We compute the derivative of the function

$$h(s)=K(x^*+sg)-\frac{2\lambda}{\sigma}\times\left(\int_T e^{\sigma\mathbf{C}(x^*+sg)(t)+\mu(t)}dt-b\right),$$

where $2\lambda/\sigma$ is the Lagrange multiplier. We take derivative with respect to s

$$h'(0) = 2\int_T x^*(t)\mathbf{C}(g)(t)dt - 2\lambda\int_T e^{\sigma\mathbf{C}(x^*)(t)+\mu(t)}\mathbf{C}(g)(t)dt. \tag{35}$$

The solution x^* satisfies $h'(0) = 0$. Since g is arbitrary, we have that x^* is a solution to (5) is equivalent to the following conditions

$$x^*(t) = \lambda e^{\sigma\mathbf{C}(x^*)(t)+\mu(t)} \text{ and } \int_T e^{\sigma\mathbf{C}(x^*)(t)+\mu(t)}dt = b. \tag{36}$$

We plug the formula of x^* in the first identity into the second identity and obtain that $\lambda = \Lambda(x^*)$ and thus x^* is a fixed point of $\mathbf{S}$. This concludes the proof. ■

Proof of Proposition 1. According to the contraction mapping theorem, the operator $\mathbf{S}$ has a unique fixed point. According to Proposition 4 whose proof is independent of the current one, this fixed point x^* is the solution to optimization problem (5). This implies that (5) has a unique solution in $\mathcal{B}$.

To prove (ii), we expand the exponents in (36) and have that

$$x^*(t) = \lambda e^{\mu(t)}(1 + O(\sigma^{\alpha-\varepsilon})) \text{ and } \int_T e^{\mu(t)}[1 + \sigma\mathbf{C}(x^*)(t)]dt + O(\sigma^{2(\alpha-\varepsilon)}) = b.$$

Based on the above two identities, we solve

$$\lambda = \frac{(1 + o(1))\kappa\sigma^{\alpha-1}}{\int_{T\times T} C(s,t)e^{\mu(s)+\mu(t)}dsdt}.$$

This yields

$$x^*(t) = (1 + o(1))\kappa\sigma^{\alpha-1}\frac{e^{\mu(t)}}{\int_{T\times T} C(s,t)e^{\mu(s)+\mu(t)}dsdt} \tag{37}$$

and

$$K^*_\sigma = (1 + o(1))\kappa^2\sigma^{2\alpha-2}\Big(\int_T\int_T C(s,t)e^{\mu(s)+\mu(t)}dsdt\Big)^{-1}.$$

■

4. Acknowledgment This research is supported in part by NSF SES-1323977, NSF, CMMI-1069064, and Army Grant W911NF-14-1-0020.

References

[1] Ahsan, S.M. 1978. Portfolio selection in a lognormal securities market. *Zeitschrift Fur Nationalokonomie-Journal of Economics* **38**(1-2) 105–118.

[2] Asmussen, Søren, Leonardo Rojas-Nandayapa. 2008. Asymptotics of sums of lognormal random variables with gaussian copula. *Statistics & Probability Letters* **78**(16) 2709–2714.

[3] Basak, S., A. Shapiro. 2001. Value-at-risk-based risk management: Optimal policies and asset prices. *Review of Financial Studies* **14**(2) 371–405.

[4] Borell, C. 1975. The Brunn-Minkowski inequality in Gauss space. *Inventiones Mathematicae* **30**(2) 207–216.

[5] Deutsch, H. P. 2004. *Derivatives and internal models*. 3rd ed. Finance and capital markets, Palgrave Macmillan, Houndmills, Basingstoke, Hampshire ; New York, N.Y.

[6] Duffie, D., J. Pan. 1997. An overview of value at risk. *The Journal of Derivatives* **4**(3) 7–49.

[7] Dufresne, D. 2001. The integral of geometric Brownian motion. *Advances in Applied Probability* **33**(1) 223–241.

[8] Foss, S., A. Richards. 2010. On sums of conditionally independent subexponential random variables. *Mathematics of Operations Research* **35** 102–119.

[9] Gao, Xin, Hong Xu, Dong Ye. 2009. Asymptotic behavior of tail density for sum of correlated lognormal variables. *International Journal of Mathematics and Mathematical Sciences* **2009**.

[10] Glasserman, P., P. Heidelberger, P. Shahabuddin. 2000. Variance reduction techniques for estimating value-at-risk. *Management Science* **46**(10) 1349–1364.

[11] Liu, J. 2012. Tail approximations of integrals of Gaussian random fields. *Annals of Probability* **40**(3) 1069–1104.

[12] Liu, J., G. Xu. 2012. Some asymptotic results of Gaussian random fields with varying mean functions and the associated processes. *Annals of Statistics* **40** 262–293.

[13] Liu, J., G. Xu. 2013. On the density functions of integrals of Gaussian random fields. *Advances in Applied Probability* **45** 398–424.

[14] Liu, J., G. Xu. 2014. On the conditional distributions and the efficient simulations of exponential integrals of Gaussian random fields. *Annals of Applied Probability* **24** 1691–1738.

[15] Tsirelson, B.S., I.A. Ibragimov, V.N. Sudakov. 1976. Norms of Gaussian sample functions. *Proceedings of the Third Japan-USSR Symposium on Probability Theory (Tashkent, 1975)* **550** 20–41.

[16] Yor, M. 1992. On some exponential functionals of Brownian-motion. *Advances in Applied Probability* **24**(3) 509–531.