Why is $\sum_{i=1} p_i < \infty$ for completely random measures?

Notes for Probabilistic Models of Discrete Data. Professor: David Blei
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Jonathan Auerbach asked in class why it is the case that $\sum_{i=1}^{\infty} p_i < \infty$ in the completely random measures (CRM) framework. Rajesh Ranganath provided a nice picture about the underlying intuition: we need this to have a properly normalized distributions. Here I complement the explanation with a 'measure theoretical' ~~police~~ proof.

1 Proposition

Consider a completely random measure $\mathcal{L}(\cdot)$ as in [ZC15]. That is, for each $A \subseteq \Omega$ consider the sum $\mathcal{L}(A) = \sum_i p_i \chi_A(\omega_i)$ of an underlying Poisson process in the product space $\mathbb{R}^+ \times \Omega$ with intensity $\nu(p, \omega)$. Also, assume it is satisfied the 'Levy-like' integrability property: for each compact $A \subseteq \Omega$:

$$\int_{\mathbb{R}^+} \int_A \min(1, p) d\nu(p, \omega) < \infty. \tag{1}$$

Then, for such compacts A,

$$S \equiv \sum_{(w_i) \in A} p_i < \infty, \qquad \text{almost surely (a.s.).}$$

Proof: We will show that S can be expressed as $S = S_1 + S_2$, with S_2 being finite a.s., and $E(S_1) < \infty$ from which it will follow, by an elementary measure theoretical property, that $S_1 < \infty$ as well.

Indeed, we can break down the above sum as the sum for the p_i that are smaller than 1, and the rest. In other words:

$$S = S_1 + S_2, \quad S_1 = \sum_{(w_i) \in A} p_i 1_{p<1}, \quad S_2 = \sum_{(w_i) \in A} p_i 1_{p \geq 1}.$$

Now, recall the Campbell theorem for Poisson processes [Kin92]: Let N be a Poisson process in a space B with intensity measure λ. Then, for a measurable real valued function f:

$$E\left(\sum_{x \in N} f(x)\right) = \int_B f(x) d\lambda(x).$$

In terms of the Campbell's theorem statement one can express $S_1 = \sum_{x \in N} f(x)$ where $N = \sum_i \delta_{(p_i, w_i)}$ (the underlying Poisson process in the product space), $x = (p, w)$ and $f(x) = p1_{p<1}$. Thus,

$$E(S_1) = \int_{\mathbb{R}^+} \int_A p1_{p<1} d\nu(p, \omega) = \int_0^1 \int_A p d\nu(p, \omega)$$

To conclude, let's re-state equation (1) as

$$\int_0^1 \int_A p d\nu(p, \omega) + \nu([1, \infty], A) < \infty. \tag{2}$$

As this sum is finite, both terms are. The first term is actually $E(S_1)$, therefore $E(S_1) < \infty$. Additionally, the fact that $\nu([1, \infty], A)$ is finite implies that a.s. S_2 contains only a finite number of terms, as the number of points in A such that $p_i > 1$ has a poisson distribution with finite rate $\nu([1, \infty], A)$.

References

[Kin92] John Frank Charles Kingman. *Poisson processes*, volume 3. Clarendon Press, 1992.

[ZC15] MengChu Zhou and Lawrence Carin. Negative binomial process count and mixture modeling. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 37(2):307–320, 2015.