Risk, Return, and Ross Recovery

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September 13, 2012

$\mathbb{P}$, $\mathbb{Q}$, and Ross Recovery

- Suppose the real-world probability measure $\mathbb{P}$ quantifies the market's belief about the frequencies of future states.
- The risk-neutral probability measure $\mathbb{Q}$ describes futures prices of Arrow Debreu securities.
- In 2011, Stephen A. Ross began circulating a working paper called "The Recovery Theorem".
- The Ross Recovery Theorem gives sufficient conditions under which $\mathbb{P}$ can be obtained from $\mathbb{Q}$.

The Ross Recovery Theorem

- Theorem 1 in Ross (2011) states that:
    - if markets are complete, and
    - if the utility function of the representative investor is state independent and intertemporally additively separable
      and:
    - if the state variable is a time homogeneous Markov process X with a finite discrete state space,
      then:
    - one can recover the real-world transition probability matrix of X from the assumed known matrix of Arrow Debreu state prices.

Vas Ist Das?

- Jiming Yu and I were intrigued by Ross's conclusion, but bothered by the restrictions on preferences that Ross made to generate it.
- In particular, it seems impossible to test whether a representative investor's utility function is additively separable or not.
- Fortunately, there is a concept in the financial literature which can be used to derive Ross's conclusion without restricting preferences.
- We exploit this concept to replace Ross's restrictions on preferences with restrictions on beliefs, which we believe to be more testable.

Enter The Numeraire Portfolio

- In 1990, John Long introduced the notion of a *numeraire portfolio*.
- A numeraire is any self-financing portfolio whose price is alway positive.
- Long showed that if any set of assets is arbitrage-free, then there always exists a numeraire portfolio comprised of just these assets.
- The defining property of this numeraire portfolio is the following surprising result:

> *If the value of the numeraire portfolio is used to deflate each asset's dollar price, then each deflated price evolves as a martingale under the real-world probability measure.*

Intuition on the Numeraire Portfolio

- The important mathematical property of the Money Market Account (MMA) is that it is a price process of bounded variation.
- When the MMA is used to finance all purchases and denominate all gains, then the $\mathbb{P}$ expected return on each asset position is usually not zero and in equilibrium would usually be ascribed to risk premium
- The important mathematical property of the numeraire portfolio is that it is a price process of *unbounded* variation.
- When the numeraire portfolio is used to finance all purchases and denominate all gains, then the mean gain on each asset is zero units of the numeraire portfolio.
- Intuitively, if one asset is expected to return more than another for whatever reason, then returns on the numeraire portfolio will be more positively correlated with the expected performer than with the laggard. As a result, neither asset outperforms the other on average, once gains are measured in units of the numeraire portfolio.
- Long's discovery of the numeraire portfolio lets one replace the abstract notion of an equivalent martingale measure $\mathbb{Q}$ with the concrete notion of portfolio value

Proof That the Numeraire Portfolio Exists

- Consider an arbitrage-free economy consisting of a default-free Money Market Account (MMA) whose balance S_{0t} grows at the stochastic short interest rate r_t, and one or more risky assets with spot prices $S_{it}, i = 1, \ldots, n$.

- These assumptions imply the existence of a martingale measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which each $r-$ discounted security price, $e^{-\int_0^t r_s ds} S_{it}$, evolves as a $\mathbb{Q}$-martingale, i.e.

$$E^{\mathbb{Q}}\left[\frac{S_{iT}}{S_{0T}}|\mathcal{F}_t\right] = \frac{S_{it}}{S_{0t}}, \qquad t \in [0, T], i = 0, 1, \ldots, n.$$

- Let M be the positive $\mathbb{P}$ martingale used to create $\mathbb{Q}$ from $\mathbb{P}$:

$$E^{\mathbb{P}}\left[\frac{M_T}{M_t}\frac{S_{iT}}{S_{0T}}|\mathcal{F}_t\right] = \frac{S_{it}}{S_{0t}}, \qquad t \in [0, T], i = 0, 1, \ldots, n.$$

- To conserve probability, M must be a $\mathbb{P}$ martingale started at one.

Proof That the Numeraire Portfolio Exists (Con'd)

- From the last slide, M is the positive $\mathbb{P}$ martingale used to create $\mathbb{Q}$ from $\mathbb{P}$:

$$E^{\mathbb{P}}\left[\frac{M_T}{M_t}\frac{S_{iT}}{S_{0T}}|\mathcal{F}_t\right] = \frac{S_{it}}{S_{0t}}, \qquad t \in [0,T], i = 0, 1, \ldots, n.$$

- Fact: If M creates $\mathbb{Q}$ from $\mathbb{P}$, then $\frac{1}{M}$ creates $\mathbb{P}$ from $\mathbb{Q}$. To conserve probability, $\frac{1}{M}$ must be a positive $\mathbb{Q}$ local martingale. Let L denote the product of the money market account S_0 and this reciprocal:

$$L_t \equiv \frac{S_{0t}}{M_t}, \qquad \text{for } t \in [0,T].$$

- L is clearly a positive stochastic process. Since L is just the product of the MMA and the $\mathbb{Q}$ local martingale $\frac{1}{M}$, L grows in $\mathbb{Q}$ expectation at the risk-free rate. As result, L is the value of some self-financing portfolio. Multiplying both sides of the top equation by M_t and substituting in L implies:

$$E^{\mathbb{P}}\frac{S_{iT}}{L_T}|\mathcal{F}_t = \frac{S_{it}}{L_t}, \qquad t \in [0,T], i = 0, 1, \ldots, n.$$

- In words, L is the value of the numeraire portfolio that Long introduced.

Intuition for Positive Continuous Price Processes

- To gain intuition, suppose each spot price S_i is positive and has continuous sample paths over time (i.e. no jumps).
- The value L of the numeraire portfolio is always positive and now it would be continuous as well.
- Let $R_{it} \equiv \frac{S_{it}}{L_t}$ be the ratio at time t of each spot price S_{it} to L_t. Itô's formula implies:

$$\frac{dR_{it}}{R_{it}} = \frac{dS_{it}}{S_{it}} - \frac{dL_t}{L_t} - \frac{dS_{it}}{S_{it}} \frac{dL_t}{L_t} + \left(\frac{dL_t}{L_t}\right)^2.$$

- The last two terms are of order dt and arise due to the unbounded variation of the sample paths of L.

Intuition for Positive Continuous Price Processes (Con'd)

- Recall that for the ratio $R_{it} \equiv \frac{S_{it}}{L_t}$, Itô's formula implies:

$$\frac{dR_{it}}{R_{it}} = \frac{dS_{it}}{S_{it}} - \frac{dL_t}{L_t} - \frac{dS_{it}}{S_{it}}\frac{dL_t}{L_t} + \left(\frac{dL_t}{L_t}\right)^2.$$

- Let r_t be the spot interest rate at time t. If:

$$E^{\mathbb{P}}\frac{dS_{it}}{S_{it}} = r_t + \frac{dS_{it}}{S_{it}}\frac{dL_t}{L_t},$$

then since L is a portfolio:

$$E^{\mathbb{P}}\frac{dL_t}{L_t} = r_t + \left(\frac{dL_t}{L_t}\right)^2,$$

and so taking expected value in the top equation implies:

$$E^{\mathbb{P}}\frac{dR_{it}}{R_{it}} = r_t - r_t = 0.$$

- Hence, the ratio $R_{it} \equiv \frac{S_{it}}{L_t}$ is a $\mathbb{P}$ local martingale. By restricting asset volatilities the stronger condition of martingality can be achieved.

Risk and Return for Each Spot Price

- Suppose we furthermore assume that the market is complete. This would arise if for example each spot price is a diffusion driven by a common state variable X.
- If we know the joint dynamics of spot prices under $\mathbb{Q}$ and if we can determine the dynamics of L under $\mathbb{Q}$, then we can determine the real world expected return on each asset since:

$$E^{\mathbb{P}} \frac{dS_{it}}{S_{it}} = r_t + \frac{dS_{it}}{S_{it}} \frac{dL_t}{L_t}.$$

- In a continuous world, the quadratic variations and covariations of martingale components under $\mathbb{P}$ are the same as under $\mathbb{Q}$.
- In a continuous setting, the bounded variation components ($=$ expected returns) under $\mathbb{P}$ arise by adding the short rate r to the covariations $\frac{dS_{it}}{S_{it}} \frac{dL_t}{L_t}$. So for continuous semi-martingales, the dynamics of each spot price under $\mathbb{P}$ would be determined.

Risk and Return for the Numeraire Portfolio

- Recall that Long showed in a continuous setting that the risk premium of the numeraire portfolio IS its instantaneous variance rate.

$$E^{\mathbb{P}}\frac{dL_t}{L_t} = r_t + \left(\frac{dL_t}{L_t}\right)^2$$

- One could not imagine a simpler relation between risk premium and risk.
- Suppose we let σ denote the instantaneous lognormal volatility of the numeraire portfolio, i.e.

$$\left(\frac{dL_t}{L_t}\right)^2 = \sigma^2 dt$$

- Then the market price of Brownian risk is σ_t.
- Our goal is now to determine this volatility, which represents both risk and reward.

Rolling Our Own

- Using Long's numeraire portfolio, we replace Ross's restrictions on the form of preferences with our restrictions on the form of beliefs.
- More precisely, we suppose that the prices of some given set of assets are all driven by a univariate time-homogenous bounded diffusion process, X.
- Letting L denote the value of the numeraire portfolio for these assets, we furthermore assume that L is also driven by X and t and that (X, L) is a bivariate time homogenous diffusion.
- We show that these assumptions determine the real world dynamics of all assets in the given set.

Our Assumptions

- We assume no arbitrage for some finite set of assets which includes a money market account (MMA).
- As a result, there exists a risk-neutral measure $\mathbb{Q}$ under which prices deflated by the MMA evolve as martingales.
- Under $\mathbb{Q}$, the driver X is a time homogeneous *bounded* diffusion:

$$dX_t = b(X_t)dt + a(X_t)dW_t, \qquad t \in [0, T].$$

  where $X_0 \in (\ell, u)$ and where W is standard Brownian motion under $\mathbb{Q}$.
- We also assume that under $\mathbb{Q}$, the value L of the numeraire portfolio solves:

$$\frac{dL_t}{L_t} = r(X_t)dt + \sigma(X_t)dW_t, \qquad t \in [0, T].$$

- We know the functions $b(x)$, $a(x)$, and $r(x)$ but not $\sigma(x)$. How to find it?

Value Function of the Numeraire Portfolio

- Recalling that X is our driver, we assume:

$$L_t \equiv L(X_t, t), \qquad t \in [0, T],$$

where $L(x, t)$ is a positive function of $x \in \mathbb{R}$ and time $t \in [0, T]$.

- Applying Itô's formula, the volatility of L is:

$$\sigma(x) \equiv \frac{1}{L(x,t)} \frac{\partial}{\partial x} L(x,t) a(x) = a(x) \frac{\partial}{\partial x} \ln L(x,t).$$

- Dividing by $a(x) > 0$ and integrating w.r.t. x:

$$\ln L(x,t) = \int^x \frac{\sigma(y)}{a(y)} dy + f(t), \text{ where } f(t) \text{ is the constant of integration.}$$

- Exponentiating implies that the value of the numeraire portfolio separates multiplicatively into a positive function $\pi(\cdot)$ of the driver X and a positive function $p(\cdot)$ of time t:

$$L(x,t) = \pi(x) p(t),$$

where: $\pi(x) = e^{\int^x \frac{\sigma(y)}{a(y)} dy}$ and $p(t) = e^{f(t)}$.

Separation of Variables

- The numeraire portfolio value function $L(x,t)$ must solve a PDE to be self-financing:

$$\frac{\partial}{\partial t}L(x,t) + \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2}L(x,t) + b(x)\frac{\partial}{\partial x}L(x,t) = r(x)L(x,t).$$

- On the other hand, the last slide shows that this value separates as:

$$L(x,t) = \pi(x)p(t).$$

- Using Bernoulli's classical separation of variables argument, we know that:

$$p(t) = p(0)e^{\lambda t},$$

and that:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in [\ell, u].$$

Regular Sturm Liouville Problem

- Recall the ODE on the last slide:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in [\ell, u].$$

- Here $\pi(x)$ and λ are unknown. This can be regarded as a regular Sturm Liouville problem.
- From Sturm Liouville theory, we know that there exists an eigenvalue λ_0, smaller than all of the other eigenvalues, and an associated positive eigenfunction, $\phi(x)$ which is unique up to positive scaling.
- All of the eigenfunctions associated to the other eigenvalues switch signs at least once.
- One can numerically solve for both the smallest eigenvalue λ_0 and its associated positive eigenfunction, $\phi(x)$. The positive eigenfunction $\phi(x)$ is unique up to positive scaling.

Value Function of the Numeraire Portfolio

- Recall that λ_0 is the known lowest eigenvalue and $\phi(x)$ is the associated eigenfunction, positive and known up to a positive scale factor.
- As a result, the value function of the numeraire portfolio is also known up to a positive scale factor:

$$L(x,t) = \phi(x)e^{\lambda_0 t}, \qquad x \in [\ell, u], t \in [0, T].$$

- As a result, the volatility of the numeraire portfolio is *uniquely* determined as:

$$\sigma(x) = a(x)\frac{\partial}{\partial x} \ln \phi(x), \qquad x \in [\ell, u].$$

- Mission accomplished! Let's see what the market believes.

Real World Dynamics of the Numeraire Portfolio

- In our diffusion setting, Long (1990) showed that the *real world* dynamics of L are given by:

$$\frac{dL_t}{L_t} = [r(X_t) + \sigma^2(X_t)]dt + \sigma(X_t)dB_t, \qquad t \geq 0,$$

  where B is a standard Brownian motion under the real world probability measure $\mathbb{P}$.

- In equilibrium, the risk premium of the numeraire portfolio is simply $\sigma^2(x)$.
- Since we have determined $\sigma(x)$ on the last slide, the risk premium of the numeraire portfolio has also been uniquely determined.
- The market price of Brownian risk is simply $\sigma(X_t)$. The function $\sigma(x)$ is now known but what about the dynamics of X?

Real World Dynamics of the Driver

- From Girsanov's theorem, the dynamics of the driver X under the real world probability measure $\mathbb{P}$ are:

$$dX_t = [b(X_t) + \sigma(X_t)a(X_t)]dt + a(X_t)dB_t, \qquad t \geq 0,$$

  where recall B is a standard Brownian motion under the real world probability measure $\mathbb{P}$.

- Hence, we know the real world dynamics of the driver X.
- We still have to determine the real world transition density of the driver X.

Real World Transition PDF of the Driver

- From the change of numeraire theorem, the Radon Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{-\int_0^T r(X_t)dt} \frac{L(X_T, T)}{L(X_0, 0)} = \frac{\phi(X_T)}{\phi(X_0)} e^{\lambda_0 T} e^{-\int_0^T r(X_t)dt},$$

  since $L(x, t) = \phi(x)e^{\lambda_0 t}$.

- Let $d\mathbb{A} \equiv e^{-\int_0^T r(X_t)dt} d\mathbb{Q}$ denote the assumed known Arrow Debreu state pricing density:

$$d\mathbb{P} = \frac{\phi(X_T)}{\phi(X_0)} e^{-\lambda_0 T} e^{-\int_0^T r(X_t)dt} d\mathbb{Q} = \frac{\phi(X_T)}{\phi(X_0)} e^{-\lambda_0 T} d\mathbb{A}.$$

- As we know the function $\frac{\phi(y)}{\phi(x)}$, the positive function $e^{-\lambda T}$, and the Arrow Debreu state pricing density $d\mathbb{A}$, we know $d\mathbb{P}$, the real-world transition PDF of X.

Real World Dynamics of Spot Prices

- Also from Girsanov's theorem, the dynamics of the $i-$th spot price S_{it} under $\mathbb{P}$ are uniquely determined as:

$$dS_{it} = [r(X_t)S_i(X_t,t)+\sigma(X_t)\frac{\partial}{\partial x}S_i(X_t,t)a^2(X_t)]dt+\frac{\partial}{\partial x}S_i(X_t,t)a(X_t)dB_t,$$

where for $x \in (\ell, u), t \in [0, T]$, $S_i(x,t)$ solves the following linear PDE:

$$\frac{\partial}{\partial t}S_i(x,t) + \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2}S_i(x,t) + b(x)\frac{\partial}{\partial x}S_i(x,t) = r(x)S_i(x,t),$$

subject to appropriate boundary and terminal conditions. If $S_{it} > 0$, then the SDE at the top can be expressed as:

$$\frac{dS_{it}}{S_{it}} = r(X_t)+\sigma(X_t)\frac{\partial}{\partial x}\ln S_i(X_t,t)a^2(X_t)]dt+\frac{\partial}{\partial x}\ln S_i(X_t,t)a(X_t)dB_t, t \geq 0.$$

- In equilibrium, the instantaneous risk premium is just $d\langle \ln S_i, \ln L\rangle_t$, i.e. the increment of the quadratic covariation of returns on S_i with returns on L.

Some Extensions to Diffusions on Unbounded Domain

- Our results thus far apply only to diffusions on bounded domains.
- Hence, our results thus far can't be used to determine whether one can uniquely determine $\mathbb{P}$ in models like Black Scholes (1973) and Cox Ingersoll Ross (1985) where the diffusing state variable X lives on an unbounded domain such as $(0, \infty)$.
- We don't yet know the general theory here, but we do know two interesting examples of it.

Example 1: Black Scholes Model for a Stock Price

- Suppose that the state variable X is a stock price whose initial value is observed to be the positive constant S_0.
- Suppose we assume or observe zero interest rates and dividends and we assume that the spot price is GBM under $\mathbb{Q}$:

$$\frac{dX_t}{X_t} = \sigma dW_t, \qquad t \geq 0.$$

- Suppose only one stock option trades and from its observed market price, we learn the numerical value of σ.
- All of our previous assumptions are in place except now we have allowed the diffusing state variable X to live on the unbounded domain $(0, \infty)$.

Example 1 (con'd): Black Scholes Model

- Recall the general ODE governing the positive function $\pi(x)$ and the scalar λ:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in (\ell, u).$$

- In the BS model with zero rates, $a^2(x) = \sigma^2 x^2$, $b(x) = r(x) = 0$, $\ell = 0$, and $u = \infty$ so we want a positive function $\pi(x)$ and a scalar λ solving the Euler ODE:

$$\frac{\sigma^2 x^2}{2}\pi''(x) = -\lambda\pi(x), \qquad x \in (0, \infty).$$

- In the class of twice differentiable functions, there are are an uncountably infinite number of eigenpairs (λ, π) with π positive. This result implies Ross can't recover here because there are too many candidates for the value of the numeraire portfolio.

- However, all of the positive functions $\pi(x)$ are not square integrable. If we insist on this condition as well, then there are no candidates for the value of the numeraire portfolio. Ross can't recover again, but for a different reason.

Example 2: Cox Ingersoll Ross (CIR) Model

- The CIR model for the short interest rate r assumes the following mean-reverting square root process under $\mathbb{Q}$:

$$dr_t = (a + br_t)dt + c\sqrt{r_t}dW_t, \qquad t \geq 0.$$

- Suppose that after calibrating to caps and floors, we get $r_0 = \frac{1}{2}$, $a = \frac{1}{2}$, $b = 0$ and $c = \sqrt{2}$ so that the risk-neutral short rate process is:

$$dr_t = \frac{1}{2}dt + \sqrt{2r_t}dW_t, \qquad t \geq 0.$$

- Suppose we choose the state variable X to be the driving SBM W but started at 1. You can check that the short rate vol process $\sqrt{2r_t}$ is just $|X|$.
- Recall the general ODE governing the positive function $\pi(x)$ & the scalar λ:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in (\ell, u).$$

- Here, $a^2(x) = 1$, $b(x) = 0$, $r(x) = \frac{x^2}{2}$, $\ell = -\infty$, and $u = \infty$ so we want a positive function $\pi(x)$ and a scalar λ solving the linear ODE:

$$\frac{1}{2}\pi''(x) - \frac{x^2}{2}\pi(x) = -\lambda\pi(x), \qquad x \in (-\infty, \infty).$$

Example 2: CIR Model (Con'd)

- Recall we want a positive function $\pi(x)$ and a scalar λ solving the ODE:

$$\frac{1}{2}\pi''(x) - \frac{x^2}{2}\pi(x) = -\lambda\pi(x), \qquad x \in (-\infty, \infty).$$

- This is a famous problem in quantum mechanics called "quantum harmonic oscillator".

- Suppose we restrict the function space to be square integrable functions on the whole real line. Then the spectrum is discrete with lowest eigenvalue $\lambda_0 = \frac{1}{2}$. The associated eigenfunction (A.K.A. ground state) is the Gaussian function

$$\phi(x) = e^{-x^2/2}, x \in (-\infty, \infty).$$

- The eigenfunction is clearly positive and moreover it is unique up to positive scaling, since the other eigenfunctions have the form:

$$\phi_n(x) = h_n(x)e^{-x^2/2}, n = 1, 2 \ldots \infty, \qquad x \in (-\infty, \infty),$$

where $h_n(x)$ is the n-th Hermite polynomial defined by the Rodrigues formula:

$$h_n(x) = e^{x^2/2} D_x^n e^{-x^2/2}, n = 1, 2, \ldots \infty.$$

Example 2: CIR Model (Con'd)

- Recall that the lowest eigenvalue is $\lambda_0 = \frac{1}{2}$ and the associated ground state is $\phi(x) = e^{-x^2/2}$, so we have established that the value function for the numeraire portfolio is:

$$L(x,t) = e^{-x^2/2} e^{t/2}, \qquad x \in (-\infty, \infty), t \geq 0.$$

- Under $\mathbb{P}$, Girsanov's theorem implies that the state variable X becomes the following Ornstein Uhlenbeck process:

$$dX_t = -X_t dt + dB_t, \qquad t \geq 0,$$

  where B is a standard Brownian motion.

- Since we still have $r_t = r(X_t) = \frac{X_t^2}{2}$, the short rate is still a CIR process, but with larger mean reversion under $\mathbb{P}$:

$$dr_t = \left(\frac{1}{2} - 2r_t\right) dt + \sqrt{2r_t} dB_t, \qquad t \geq 0.$$

- In this example, Ross recovery fully succeeded and moreover we used his model!

Summary

- We highlighted Ross's Theorem 1 and propose an alternative preference-free way to derive the same financial conclusion.
- Our approach is based on imposing time homogeneity on the real world dynamics of the numeraire portfolio when it is driven by a bounded time homogeneous diffusion.
- We showed how separation of variables allows us to separate beliefs from preferences.

Extensions

- We suggest the following extensions for future research:
  1. Extend this work to two or more driving state variables.
  2. Explore sufficient conditions under which Ross recovery is possible on unbounded domains.
  3. Explore the extent to which Ross's conclusions survive when the driving process X is generalized into a semi-martingale.
  4. Explore what restrictions on $\mathbb{P}$ can be obtained when markets are incomplete.
  5. Implement and test.