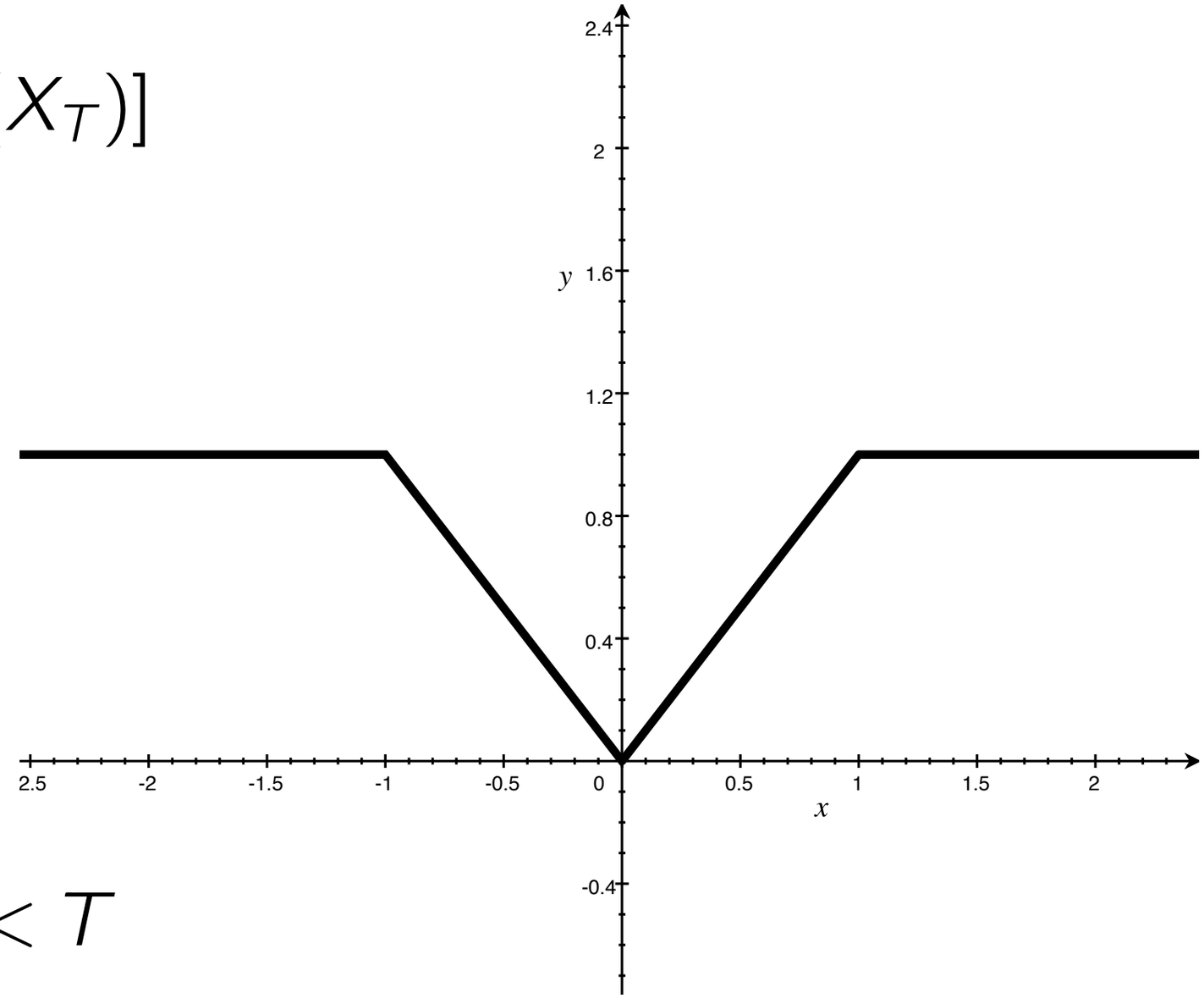
“Facelifting” in Mathematical Finance

Gordan Žitković
Department of Mathematics
The University of Texas at Austin

$$v(t,x) = \sup_\pi \mathbb{E}^{(t,x)}[g(X_T)]$$

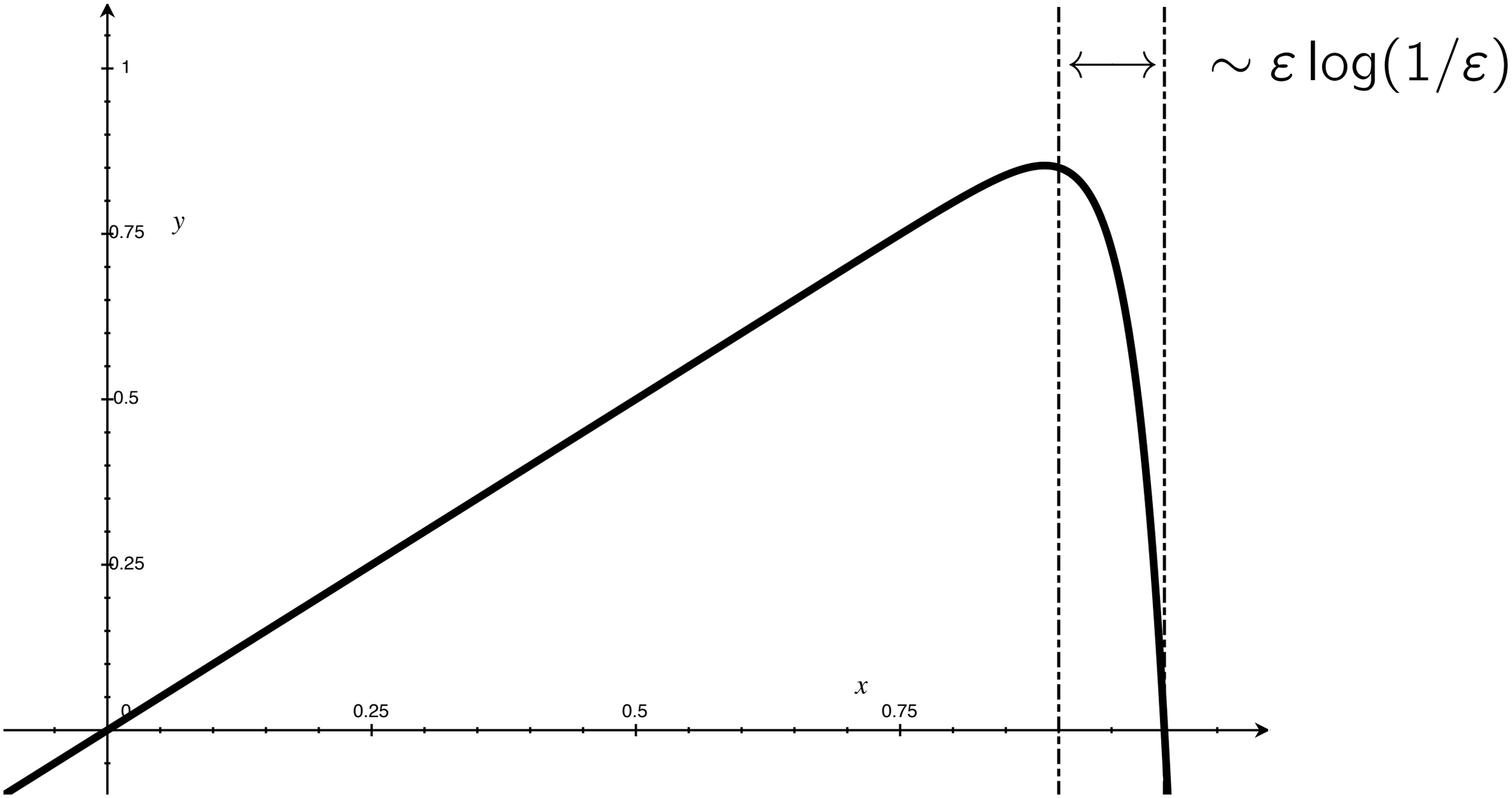
$$dX_t = \pi_t \, dB_t$$

$$g(x) = 1 \wedge |x|$$



$$v(t,x) = \begin{cases} 1, & t < T \\ g(x), & t = T \end{cases}$$

A **facelift** (boundary layer) appears - $v(T-,\cdot) \neq v(T,\cdot)$.



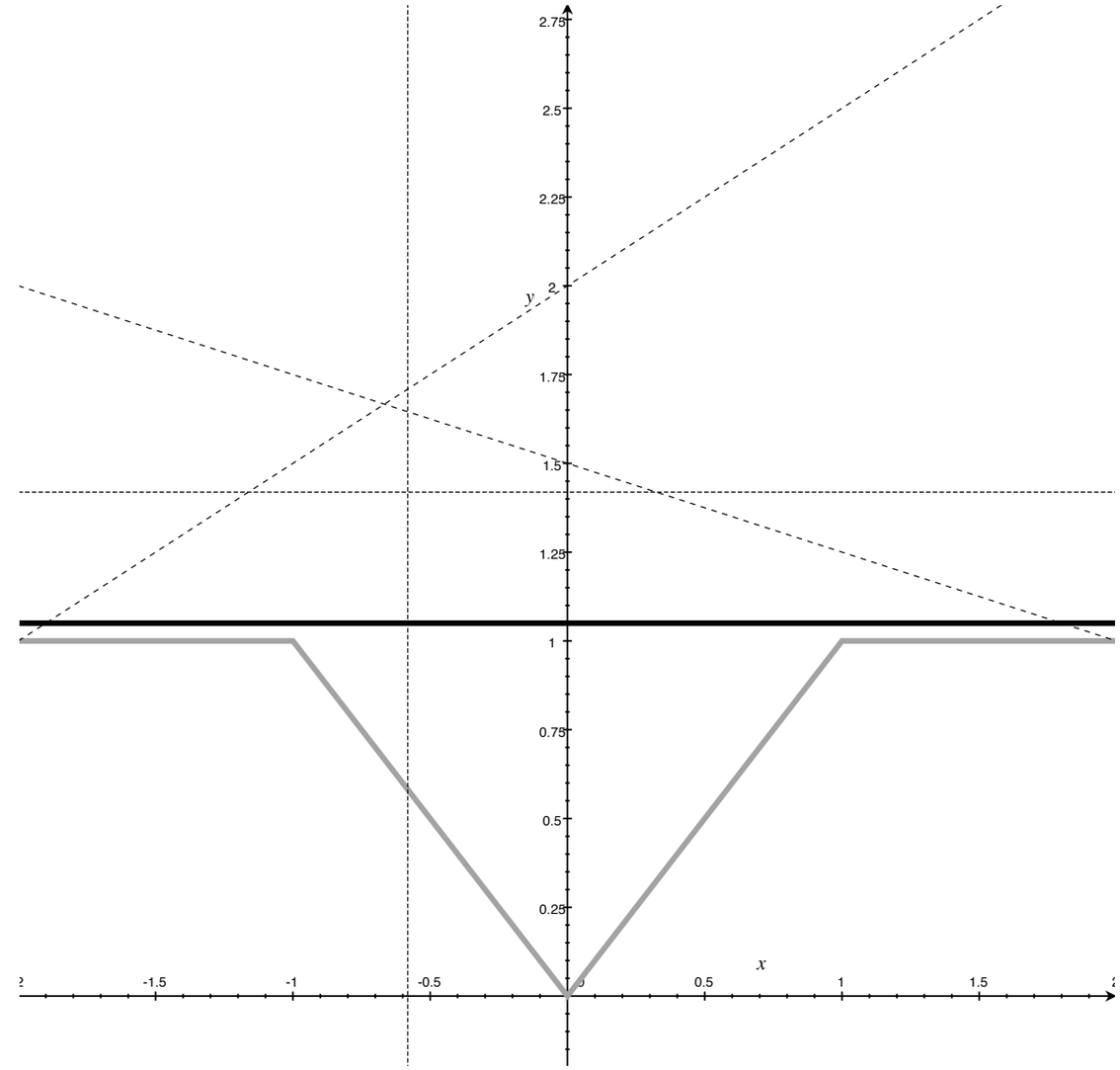
$$-\varepsilon u''(x) + u'(x) = 1, \;\; u(0) = u(1) = 0$$

The (formal) HJB equation

$$u_t + \mathcal{H}(t, x, Du, D^2u) = 0$$
$$u(T, \cdot) = g(\cdot).$$

In our example

$$\mathcal{H}(t, x, p, P) = \sup_{\pi} \tfrac{1}{2}\pi^2 P$$
$$= \begin{cases} 0, & P = 0 \\ +\infty, & P \neq 0 \end{cases}$$



The "Hamiltonian" test:

Definition. f is called $\mathcal{H}$-**finite** if $\mathcal{H}(\cdot, Df, D^2f) < \infty$.

Theorem. $v(T-, \cdot)$ is the least $\mathcal{H}$-finite envelope of φ.

η - factor process, say Feller. $(\mathbb{P}^{(t,\eta)})_{(t,\eta)}$ - Markov family.

$$dS_t = S_t\Big(\theta(\eta_t)\,dt + dB_t\Big)$$

The superhedging price:

$$p(t,\eta) = \inf\Big\{x \in \mathbb{R} \,:\, \exists\, \pi \in \mathcal{A}^{(t,\eta)},\ x + \int_t^T \pi_u\, dS_u \geq g(\eta_T),\ \mathbb{P}^{(t,\eta)} - a.s.\Big\},$$

of the contingent claim $g(\eta_T)$.

A dual representation:

$$p(t,\eta) = \sup_{\mathbb{Q}\in\mathcal{M}^{(t,\eta)}} \mathbb{E}^{\mathbb{Q}}[g(\eta_T)]$$

Question. Is there a facelift, i.e., do we have

$$p(T-,\cdot) = p(T,\cdot)?$$

Example (No). Complete market ($\mathcal{M}^{(t,\eta)}$ a singleton):

$$p(T-,\eta) = p(T,\eta) = g(\eta).$$

Example (Yes). Take $\eta = W, \theta = 0$, and $g(x) = 1 \wedge |x|$:

$$p(t,\eta) = \sup_{\nu \in \mathcal{P}^\infty} \mathbb{E}^{(t,\eta)}[g(W_T + \int_t^T \nu_u \, du)].$$

Using $\nu^{(n)} = n$, we conclude $p(t,\eta) = 1$, for $t < T$.

Problem (partly open). Does the limit $p(T-,\eta)$ exist, where

$$p(t,\eta) = \sup_{\nu \in \mathcal{P}^\infty} \mathbb{E}^{(t,\eta)}[g(\eta_T)],$$

with dynamics $d\eta_t = (\mu(\eta_t) + \nu_t \sigma(\eta_t)) \, dt + \Sigma(\eta_t) \, dW_t$?

Facelifts in Mathematical Finance:

Cvitanić, Karatzas (93) - superreplication in incomplete markets

El Karoui, Quenez (95) - Doob-type decomposition

Cvitanić, Pham, Touzi (99) - superreplication

Broadie, Cvitanić, Soner (98) - portfolio constraints

Bouchard, Touzi (00) - transaction costs

Soner, Touzi (00,04) - gamma constraints

Soner Touzi (02) - target problems, Hamiltonian test

Guasoni, Rasonyi, Schachermayer (08) - transaction costs

$U : (-a, \infty) \to \mathbb{R}$ - a utility function,

$$u(B) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U\big(\int_0^T \pi_u \, dS_u + B\big)\right].$$

Definition. A number p^i is the **indifference price of** B if

$$u(0) = u(B - p^i).$$

Definition. A number p^m is the **marginal (Davis) price** of B if

$$u(0) \geq u(qB - qp^m), \text{ for all } q.$$

The dual problem (where $\tilde{U}(y) = \sup_x (U(x) - xy)$):

$$\tilde{u}(y, B) = inf_{\mathbb{Q} \in \mathcal{M}^{ba}} \left(\mathbb{E}[\tilde{U}(y \tfrac{d\mathbb{Q}^r}{d\mathbb{P}})] + y \langle \mathbb{Q}, B \rangle \right).$$

$\tilde{u}(y, B) = \inf_{\mathbb{Q}\in\mathcal{M}^{ba}} \Big(\mathbb{E}[\tilde{U}(y\frac{d\mathbb{Q}^r}{d\mathbb{P}})] + y\langle\mathbb{Q}, B\rangle\Big)$

Theorem (Owen, Ž). We have

$$p^i(B) = \inf_{\mathbb{Q}\in\mathcal{M}^{ba}} \Big(\langle\mathbb{Q}, B\rangle + \alpha(\mathbb{Q})\Big),$$

where the penalty functional α is given by

$$\alpha(\mathbb{Q}) = \inf_y \tfrac{1}{y}\Big(\mathbb{E}[\tilde{U}(y\frac{d\mathbb{Q}^r}{d\mathbb{P}})] - \inf_z \tilde{u}(z, 0)\Big).$$

Theorem (Larsen, Owen, Ž). p^m is a marginal price if and only if there exists a minimizer $(y^*, \mathbb{Q}^*)$ of $\inf_y \tilde{u}(y, 0)$ such that

$$p^m = \langle\mathbb{Q}^*, B\rangle.$$

Question: What can we say about the facelift in the (Markovian edition) of the dual problem?

Consequences: The size of the facelift is directly related to:

- Existence of countably additive dual optimizers.
- Uniqueness of marginal utility-based prices.
- Numerical treatment of the problem.
- PDE characterization (as we will see soon).

Let $(\mathbb{P}^{(t,\eta)})_{t,\eta}$ be a Feller family, η Polish-space valued, $g \in C_b$.

$$v(t,\eta,z) = \inf_{Z \in \mathcal{Z}_b^{(t,\eta)}(z)} \mathbb{E}^{(t,\eta)}\left[V(Z_T) + Z_T g(\eta_T)\right].$$

Proposition (Larsen, Ž, 2011) $v = \tilde{u}$.

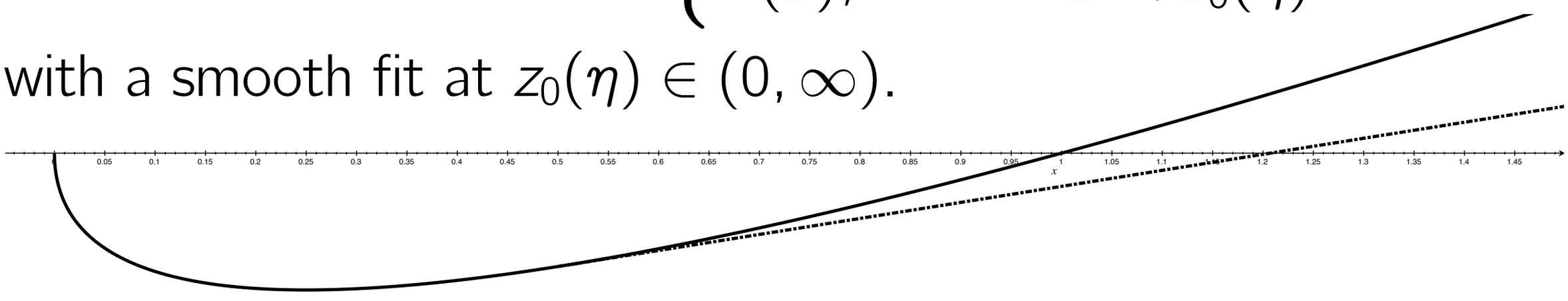
Theorem. (Larsen, Ž) Define

$$\Pi(t,\eta) = \inf_{Z \in \mathcal{Z}_b^{(t,\eta)}(1)} \mathbb{E}^{(t,\eta)}\left[Z_T g(\eta_T)\right].$$

If $\Pi(T-,\cdot)$ exists, then so does $v(T-,\cdot)$ and is given by

$$v(T-,\eta,z) = \begin{cases} z\Pi(T-,\eta), & z \geq z_0(\eta) \\ V(z), & z < z_0(\eta) \end{cases}$$

with a smooth fit at $z_0(\eta) \in (0,\infty)$.



PDE Characterization? We analyze a special case (no real l.o.g.):

$$dS_t = t + dB_t. \qquad \begin{aligned} dZ_t^\nu &= -Z_t^\nu \Big(dB_t + \nu_t \, dW_t \Big). \\ d\eta_t &= \qquad\qquad\qquad\qquad dW_t \end{aligned}$$

$$v(t, \eta, z) = \inf_\nu \mathbb{E}^{(t,\eta,z)} \left[V(Z_T^\nu) + Z_T^\nu g(\eta_T) \right]$$

Formal HJB: $0 = v_t + \inf_\nu \left(\frac{1}{2} z^2 (1 + \nu^2) v_{zz} + \frac{1}{2} v_{\eta\eta} - z \nu v_{z\eta} \right)$,

i.e.,
$$0 = v_t + \frac{1}{2} \Big(z^2 v_{zz} + v_{\eta\eta} \Big) - \frac{v_{z\eta}}{2 v_{zz}^2}. \qquad \text{(PDE)}$$

Question. Is the value function v characterized by (PDE)?

An approximation:

$$v^{(n)}(t,\eta,z) = \inf_{|\nu|\le n} \mathbb{E}^{(t,\eta,z)}\left[V(Z_T^\nu) + Z_T^\nu g(\eta_T)\right].$$

Proposition.

1. Each v_n is a classical solution to

$$\begin{cases} v_t^{(n)} + \inf\limits_{|\nu|\le n}\left(\frac{1}{2}z^2(1+\nu^2)v_{zz}^{(n)} + \frac{1}{2}v_{\eta\eta}^{(n)} - z\nu v_{z\eta}^{(n)}\right) = 0, \\ v^{(n)}(T,\eta,z) = V(z) + zg(\eta), \end{cases}$$

   and $v^{(n)} \to v$, locally uniformly on $[0,T) \times \mathbb{R} \times (0,\infty)$.

2. (PDE) + facelifted condition $\to$ solved by v

3. (PDE) + original condition $\to$ solved by **another function**.

Hamiltonian: $\mathcal{H}(\cdot, Du, D^2u) = \mathcal{L}u + \inf_\nu \mathcal{N}^\nu u$

$$\mathcal{L}u = \tfrac{1}{2}z^2 u_{zz} + \tfrac{1}{2}u_{\eta\eta} \qquad \mathcal{N}^\nu u = \tfrac{1}{2}\nu^2 z^2 u_{zz} + \nu z u_{z\eta}.$$

The Hamiltonian test does not apply:

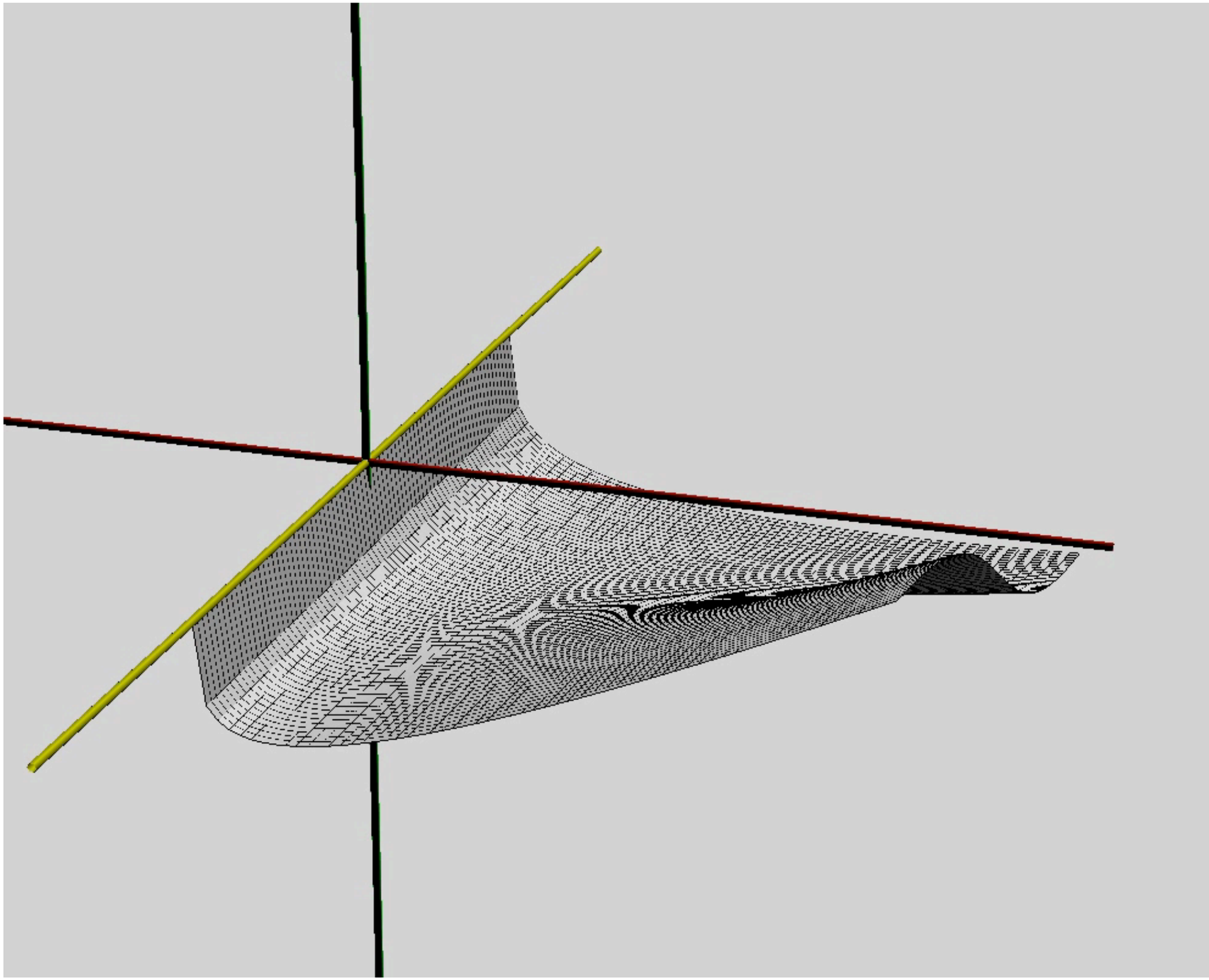
$$(\mathcal{L}+\mathcal{N}^\nu)(V(z)+zg(\eta)) \geq \tfrac{1}{2}\Big(z^2V''(z)+zg''(\eta)\Big) - \tfrac{1}{2}\tfrac{g'(\eta)}{(V''(z))^2} > -\infty.$$

Integration over space: $\langle \cdot, \cdot \rangle = \int_z \int_\eta$. Smooth test function ϕ

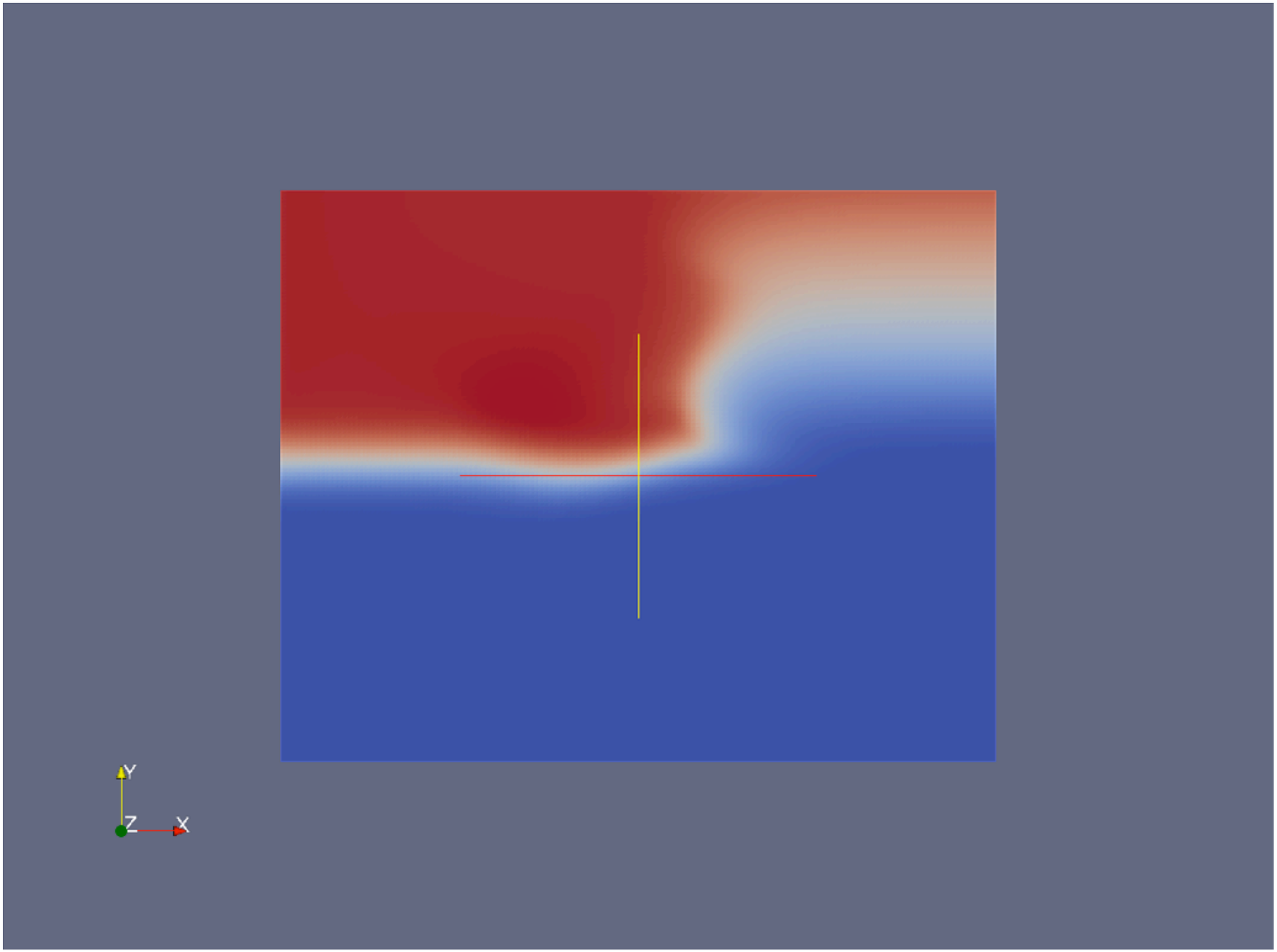
$$\langle v(T-\varepsilon, \cdot), \varphi\rangle - \langle v(T, \cdot), \varphi\rangle$$
$$= \int_{T-\varepsilon}^T \langle v, \mathcal{L}^*\varphi\rangle + \int_{T-\varepsilon}^T \langle \inf_\nu \mathcal{N}^\nu v, \varphi\rangle.$$

Leads to a test of the form:

$$\int_z \inf_\nu \mathcal{N}^\nu V(T, z, \eta)\varphi(z)\, dz < \infty, \ \forall \varphi \in \Phi^{\mathcal{L}}.$$



A Numerical Experiment - Naively



$$V(t, z, \eta) = V^0(t, z) + z\phi(t, z, \eta).$$

(FreeFem++ and VTK)

Conclusions.

Somewhat unexpectedly, utility-based pricing comes with a facelift.

Moreover, it is crucial for the proper PDE characterization.

In fact, the formal approach produces the wrong function.

Dual optimizers are almost never equivalent martingale measures.

Poor local approximation by deterministic controls close to T.

The Hamiltonian test does not apply.

THE END.