Functional Ito Calculus and Financial Applications

We first briefly review the functional Ito calculus, which is a framework that models path dependency and accounts for the fact that most frequently the relationship between a cause and an effect is not instantaneous; it depends on the history of the process.

Functional Ito calculus review

The functional Ito formula is a functional extension of the classical Ito formula, with a similar structure but with a different interpretation of the derivatives. It provides an extension of the Feynman-Kac formula to the path dependent case, in two different directions: the dynamics may be path dependent and the function may be path dependent. It provides an explicit expression of the integrand in the Martingale Representation Theorem, providing a smoother alternative to the Clark-Ocone representation. It also gives a functional or path dependent PDE (PPDE) for the pricing of path dependent options.

Lie bracket and path dependency

The functional time and space derivatives are central to the theory and in general they do not commute. Their Lie bracket essentially captures the path dependency. It appears in the Taylor expansion of a functional with respect to a path, and in the loss of martingality of the Delta of an option, even in the Bachelier case, among others.

Applications to volatility hedging

The functional Ito calculus gives the impact of a perturbation coefficient on the price of a path dependent option. This gives in turn the optimal hedge of a path dependent option with a portfolio of European option that has the same average conditional second derivative. It also provides the residual risk once the optimal hedge has been applied.

Application to super-replication

Under some conditions we obtain a refinement of the increasing process in the Kramkov optional decomposition. The increasing process comes from gains due that the worst case that was considered is not realized. This increasing process can be further decomposed into two increasing processes, one due to the functional time derivative of the lower bound process and the second one to its functional second order space derivatives. Furthermore, the two increasing process are disjoint, in the sense that they cannot increase both at the same time.