Upgrading your passport*

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The article of Hyer, Lipton-Lifschitz and Pugachevsky published in Risk No. 9/1997 introduced a contract known now as the *passport option*. This contract allows the holder of the option to switch at any time during the life of the option between short and long positions in an underlying asset. The absolute value of the number of shares held must always be less than or equal to 1. The holder accumulates gains and losses resulting from his trading strategy, and at the expiration of the option he gets the call option payoff with strike 0 on his final wealth, i.e., he keeps any gain from trading and is forgiven any loss.

The passport option has some undesirable properties. It is expensive and if the holder wants to follow the optimal strategy, he must switch his position in the underlying asset every time his account value crosses zero. If he starts at zero, in the probabilistic model his account value will cross infinitely many times in an arbitrarily small interval of time. In reality one can switch only a finite number of times, but the theoretical result suggests that the holder of the option would have a good incentive to switch many times, thus imposing large operational costs. Moreover, the switches are between extremes; for the passport option, it is never optimal to have a zero position in the underlying asset.

This fact leads us to suggest new variations of the passport option with more desirable properties. Two unexplored types are options where the holder can switch between a long and a zero position (which we call *vacation call*) or where the holder can switch between a short and a zero position (which we call *vacation put*). In contrast to the passport option, the probability that the optimal holder of vacation option would switch at all during the lifetime of the contract is relatively small. These options shall be seen to generalize the American call and the American put. Switching to a zero position in the vacation option contract does not terminate it; unlike the case of the American option, the holder of the vacation option has the right to switch back to a long (or a short) position. The vacation call and vacation put give more choices to the holder than the American call and American put, but for almost the same price. This is similar to the relationship between American and European options on non-dividend paying asset; the American option gives more optionality than the European option for the same or nearly the same price.

In this article, we study *options on a traded account*, a generalization of the passport option, where the holder of the option can switch between positions α_t and β_t ($\alpha_t \leq \beta_t$) in the underlying asset. When $\alpha_t = -1$ and $\beta_t = 1$, we have the previously studied passport option. When $\alpha_t = 0$ and $\beta_t = 1$, we have the vacation call; when $\alpha_t = -1$ and $\beta_t = 0$, we have the vacation put. The case $\alpha_t = \beta_t = 1$ shall be seen to reduce to the European call, the case $\alpha_t = \beta_t = -1$ reduces to European put. European, American, passport, vacation and even Asian options are just special cases of options on a traded account.

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There are several articles which address the pricing of passport options, see Hyer et al. [7], Andersen et al. [2], Henderson et al. [6], Ahn et al. [1] and Nagayama [10]. More recently, Lipton found connections of pricing passport options to pricing lookback and Asian options in [9].

Formulating the problem

Suppose that the underlying asset evolves under the risk neutral measure according to the equation

$$dS_t = S_t(\nu dt + \sigma dW_t), \tag{1}$$

where $\nu = r$ is the interest rate when the underlying asset is a stock or $\nu = r_d - r_f$ is the difference between the domestic and the foreign interest rate when the underlying asset is an exchange rate, and σ is the volatility of the underlying asset. Denote the option holder's trading strategy by q_t, the number of shares held at time t. The strategy q_t cannot exceed the contractual constraint $q_t \in [\alpha_t, \beta_t]$, where $\alpha_t \leq \beta_t$. In our model the value of the option holder's account corresponding to the strategy q satisfies

$$\begin{aligned} dX_t^q &= q_t dS_t + \mu(X_t^q - q_t S_t)dt \qquad (2) \\ X_0^q &= X_0. \end{aligned}$$

This represents a trading strategy in the money market and the underlying asset, where X_0 is the initial wealth and μ is the interest rate corresponding to reinvesting the cash position $X_t^q - q_t S_t$ (possibly different from the risk-neutral interest rate r). The trading strategy is self-financing when $\mu = r$. The holder of the option will receive at time T the payoff $[X_T^q]^+$. The objective of the seller of the option, who makes this payment, is to be prepared to hedge against all possible strategies of the holder of the option. Therefore the price of this contract at time t should be the maximum over all possible strategies q_u of the discounted expected value under the risk-neutral probability $\mathbb{P}$ of the payoff of the option, i.e.,

$$V^{[\alpha,\beta]}(S_0, X_0) = \max_{q_u \in [\alpha,\beta]} e^{-rT}\mathbb{E}[X_T^q]^+. \tag{3}$$

Computation of the expression in (3) is a problem of stochastic optimal control. We can obtain closed form solutions for the self-financing strategies when $\nu = \mu = r$. This is the case we consider from now on.

Relationships

The vacation call and the vacation put are cheaper alternatives to the passport option. These resemble the American call and American put, respectively. To see these connections, we note that a European call gives the holder the payoff $(S_T - K)^+$, where K is the strike. In terms of an option on a traded account, one may write

$$\begin{aligned} dX_t &= rX_t dt + q_t \sigma S_t dW_t, \quad q_t \equiv 1, \qquad (4) \\ X_0 &= S_0 - e^{-rT}K, \qquad (5) \end{aligned}$$

which represents the value of the traded account provided the option holder is always long one share of the stock. The solution of (4), (5) is $X_t = S_t - e^{-r(T-t)}K$, and therefore, $X_T = S_T - K$. The European call is an option on a traded account with the constraint $q_t \equiv 1$. We have an analogous situation with European put, which gives the holder the payoff $(K - S_T)^+$, where K is the strike. This is the same as an option on a traded account with the constraint of being always short in the underlying asset and with the initial wealth $X_0 = e^{-rT}K - S_0$.

Figure 1 illustrates the relationship among the value EC of a European call, the value VC of a vacation call, and the value PP of a passport option as functions of volatility of the underlying stock when $T = 1$, $S_0 = 100$. In this figure, $X_0 = 0$, which means that the call is struck at-the-money if $r = 0$. Because of the put-call parity, prices of calls are the same as prices of puts when $X_0 = 0$. Notice that for all reasonable volatilities the prices of the vacation call and European call are indistinguishable. This means that the insurance premium for a dynamic strategy is almost the same as for a static strategy. The passport option value increases rapidly with increasing volatility, and the holder of this option must switch positions often to realize this extra value.

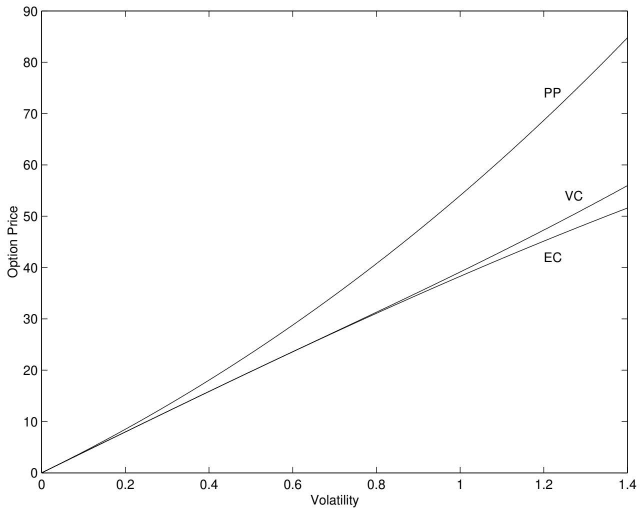

Figure 1: Prices of European Call (EC), Vacation Call (VC) and Passport Option (PP)

Suppose we give the owner of an option on a traded account the right to switch from $q_t = -1$ to $q_t = 0$ exactly once during the life of the option. If he switches at time t, he will at that time have $X_t = e^{rt}(X_0 + S_0) - S_t$, which he then invests in the money market until time T, so that $X_T = e^{rT}(X_0 + S_0) - e^{r(T-t)}S_t$. If the holder of American put with the strike K exercises at time t, he gets $X_t = (K - S_t)$ if $K > S_t$ and at date T has $X_T = e^{r(T-t)}(K - S_t)$. Equating the date T values, we get the condition

$$X_0 = e^{-rt}K - S_0. \tag{6}$$

Therefore both vacation options and American options generalize the concept of the European option, but in a different way. Since the exercise day t is not specified in advance for the case of American option, we get different solutions for X_0 depending on the exercise day. This is caused by the fact that the holder of the American option does not receive the interest on his traded account while he is in the market.

Since the price of the options on a traded account is increasing in initial wealth X_0, by taking the maximal possible value of $X_0 = K - S_0$ in (6), we get a bound on the value of the American put

$$AP(S_0, K) \leq VP(S_0, K - S_0).$$

However if we take the minimal possible value of $X_0 = e^{-rT}K - S_0$ in (6), we often get the situation when the price of the American put is significantly higher for all reasonable volatilities. As seen above, with this initial wealth, the vacation put directly generalizes the European put with strike K. Figure 2 illustrates the difference of the price of American put and the price of vacation put when $S_0 = 100$, $K = 100$, $r = 5\%$, $T = 1$ and $X_0 = e^{-rT}K - S_0$ as a function of the volatility. Notice that for all reasonable volatilities, the price of American put is higher by \$ 0.50 than the price of the vacation put. The vacation put is more expensive only for extremely large volatilities, higher than 90%.

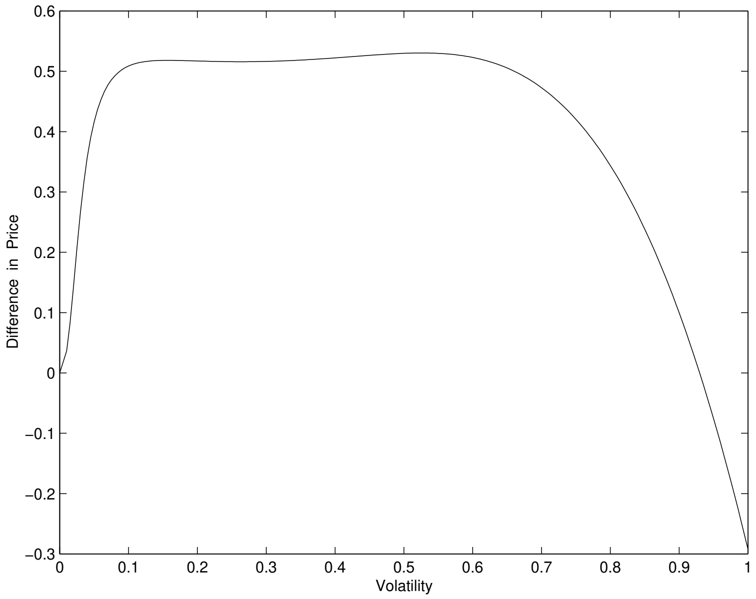

Figure 2: Difference between the price of American put and the vacation put

Options on a traded account also represent Asian options. Notice that $d(tS_t) = tdS_t + S_t dt$, or equivalently, $TS_T = \int_0^T tdS_t + \int_0^T S_t dt$. After dividing by T and rearranging the terms we get

$$\bar{S}_T := \frac{1}{T}\int_0^T S_t dt = \int_0^T \left(1 - \frac{t}{T}\right) dS_t + S_0. \tag{7}$$

In the terminology of the option on a traded account, the Asian call payoff $(\bar{S}_T - K)^+$ could be achieved by taking $q_t = 1 - \frac{t}{T}$ and $X_0 = S_0 - K$ and where the traded account evolves according to the equation

$$dX_t = \left(1 - \frac{t}{T}\right) dS_t,$$

i.e., no interest is added or charged to the traded account ($\mu = 0$ in (2)). We have then $X_T = \bar{S}_T - K$. Thus the average of the stock price could be achieved by a selling off one share of stock at the constant rate $\frac{1}{T}$ shares per unit time. Based on this relationship, a new PDE method for pricing arithmetic average Asian options has been implemented in [12].

Choice of strategy and the value of the option

It turns out that in general the optimal strategy as a function of the traded account value and the stock price is

$$q_t^{opt} = \alpha_t I_{\{(X_t^{q^{opt}}) \geq \frac{\alpha_t + \beta_t}{2} S_t\}} + \beta_t I_{\{(X_t^{q^{opt}}) < \frac{\alpha_t + \beta_t}{2} S_t\}}, \tag{8}$$

where I denotes the indicator function. In [11], closed form solutions are provided for the prices of the option on a traded account for any general constraints of the type $\alpha_t \equiv \alpha$ and $\beta_t \equiv \beta$. In particular, we have that the price of the passport option ($\alpha = -1$, $\beta = 1$) is

$$PP(t,s,x) = \tfrac{1}{2}\Big(x + (s+x)N(d_+) - sN(d_-) - s\bar{\sigma}d_- N(-d_-) + \tfrac{s\bar{\sigma}}{\sqrt{2\pi}} \exp\Big(-\tfrac{d_-^2}{2}\Big)\Big) \tag{9}$$

for $x \geq 0$, where

$$d_\pm = \tfrac{1}{\bar{\sigma}} \log(1 + \tfrac{x}{s}) \pm \tfrac{\bar{\sigma}}{2}, \quad \bar{\sigma} = \sigma\sqrt{T-t},$$

and

$$PP(t,s,x) = \tfrac{1}{2}\Big(x + (s-x)N(\epsilon_+) - sN(\epsilon_-) - s\bar{\sigma}\epsilon_- N(-\epsilon_-) + \tfrac{s\bar{\sigma}}{\sqrt{2\pi}} \exp\Big(-\tfrac{\epsilon_-^2}{2}\Big)\Big) \tag{10}$$

for $x \leq 0$, where

$$\epsilon_\pm = \tfrac{1}{\bar{\sigma}} \log(1 - \tfrac{x}{s}) \pm \tfrac{\bar{\sigma}}{2}, \quad \bar{\sigma} = \sigma\sqrt{T-t}.$$

The optimal strategy for the passport option is $q_t^{opt} = -\mathrm{sgn}\left(X_t^{q^{opt}}\right)$, i.e., be short when ahead and be long when behind. The optimal strategy and the price of passport option was previously derived by Hyer, Lipton-Lifschitz and Pugachevsky [7], Andersen, Andreasen and Brotherton-Rattcliffe [2] and by Henderson and Hobson [6].

For the vacation call, the optimal strategy is $q_t^{opt} = I_{\{X_t^{q^{opt}} < \frac{1}{2}S_t\}}$. The value of the vacation call option is given by

$$VC(t,s,x) = \frac{1}{4}\left(1 - \overline{\sigma}d_-\right) sN(-d_-) + \frac{1}{4\sqrt{2\pi}}\overline{\sigma}s \exp\left(-\tfrac{1}{2}d_-^2\right) + xN(d_+) \tag{11}$$

for $x \geq \frac{1}{2}s$, where $d_\pm = \frac{1}{\overline{\sigma}} \log\left(\frac{4x}{s}\right) \pm \frac{\overline{\sigma}}{2}$, and

$$\begin{aligned} VC(t,s,x) &= \frac{1}{4\sqrt{2\pi}}\overline{\sigma}s \exp\left(-\tfrac{1}{2}e_{+-}^2\right) - (x-s)N(-e_{++}) \\ &\quad -\frac{1}{4}(\overline{\sigma}e_{+-} + 1)sN(-e_{+-}) + (x-s)N(-e_{-+}) + sN(-e_{--}) \end{aligned} \tag{12}$$

for $x \leq \frac{1}{2}s$, where $e_{\pm\pm} = \frac{1}{\overline{\sigma}} \log\left(2 - \frac{2x}{s}\right) \pm \frac{1}{\overline{\sigma}} \log 2 \pm \frac{\overline{\sigma}}{2}$.

For the vacation put, the optimal strategy is $q_t^{opt} = -I_{\{X_t^{opt} \geq -\frac{1}{2}S_t\}}$. The value of the vacation put option is given by

$$\begin{aligned} VP(t,s,x) &= \frac{1}{4\sqrt{2\pi}}\overline{\sigma}s \exp\left(-\tfrac{1}{2}d_{+-}^2\right) + (x+s)\left[N(-d_{++}) + N(d_{-+})\right] \\ &\quad -\frac{1}{4}(\overline{\sigma}d_{+-} + 1)sN(-d_{+-}) - sN(d_{--}) \end{aligned} \tag{13}$$

for $x \geq -\frac{1}{2}s$, where $d_{\pm\pm} = \frac{1}{\overline{\sigma}} \log\left(\frac{2x}{s} + 2\right) \pm \frac{1}{\overline{\sigma}} \log 2 \pm \frac{\overline{\sigma}}{2}$, and

$$VP(t,s,x) = \frac{1}{4\sqrt{2\pi}}\overline{\sigma}s \exp\left(-\tfrac{1}{2}e_-^2\right) + \frac{1}{4}(1 - \overline{\sigma}e_-)sN(-e_-) + xN(-e_+), \tag{14}$$

for $x \leq \frac{1}{2}s$, where $e_\pm = \frac{1}{\overline{\sigma}} \log\left(-\frac{4x}{s}\right) \pm \frac{\overline{\sigma}}{2}$.

The seller's hedge

The seller of an option on a traded account wants to ensure he can pay off the obligation implied by his short position, regardless of the strategy q used by the option holder. To do this, at each time t he must know the option holder's position q_t at that time, but does not need to know the holder's future strategy. In practice, the seller of the option manages the option holder's account. We denote by p_t the number of shares of the underlying asset held by the seller at time t and by c_t the (nonnegative) rate at which the seller consumes at time t. Then the value of the seller's portfolio begins at

$$X_0^p = V^{[\alpha,\beta]}(0, S_0, X_0) \tag{15}$$

and evolves according to the equation (cf. (2))

$$dX_t^p = rX_t^p dt + p_t \sigma S_t dW_t - c_t dt. \tag{16}$$

With X_t^p, $0 \le t \le T$ determined by (15), (16), and with

$$\begin{aligned} p_t &= V_s^{[\alpha,\beta]}(t, S_t, X_t^q) + q_t V_x^{[\alpha,\beta]}(t, S_t, X_t^q) \\ c_t &= \frac{1}{2}\sigma^2 S_t^2 \left(2(q_t^{opt} - q_t)V_{sx}^{[\alpha,\beta]} + \left((q_t^{opt})^2 - q_t^2\right) V_{xx}^{[\alpha,\beta]}\right) \ \ge\ 0, \end{aligned}$$

we have $X_T^p = (X_T^q)^+$. Therefore the writer of the option may consume at a positive rate whenever the holder of the option behaves suboptimally.

Conclusion

In this paper, we noticed that European, American, passport, vacation and Asian options are special cases of options on a traded account. Because of their desirable properties for both the holder and the seller, vacation options should be considered as an alternative to European or American options.

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