Maximum Drawdown and Directional Trading

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Abstract

In this paper, we introduce new techniques how to control the maximum drawdown (MDD). One can view the maximum drawdown as a contingent claim, and price and hedge it accordingly as a derivative contract. Trading drawdown contracts or replicating them by hedging would directly address the concerns of portfolio managers who would like to insure the market drops. Similar contracts can be written on the maximum drawup (MDU). We show that buying a contract on MDD or MDU is equivalent to adopting a momentum trading strategy, while selling it corresponds to contrarian trading. We also discuss more complex products, such as plain vanilla options on the maximum drawdown or drawup, or related barrier options on MDD and MDU which we call crash and rally options, respectively.

1 Drawdowns and Drawups

Suppose that we have an underlying asset whose price process at time t is given by S_t. For example, the price process could be a stock price, index, interest rate or exchange rate. Denote by M_t its running maximum up to time t:

$$M_t = \max_{u\in[0,t]} S_u. \tag{1}$$

Drawdown D_t is defined as the drop of the asset price from its running maximum:

$$D_t = M_t - S_t. \tag{2}$$

Maximum drawdown MDD_t is defined as the maximal drop of the asset price from its running maximum over a given period of time:

$$MDD_t = \max_{u\in[0,t]} D_u. \tag{3}$$

Assume that you have an investor who enters the market at a certain point and leaves it at some following point within a given time period. Maximum drawdown measures the worst loss of such an investor, meaning that he buys the asset at a local maximum and sells it at the subsequent lowest point, and this drop is the largest in the given time period. This represents the worst period for holding this asset and could be written mathematically as:

$$MDD_t = \sup_{\substack{q_u\in\{0,1\} \\ \text{one switch } (0\to 1) \text{ and } (1\to 0)}} \int_0^t (-q_u)dS_u. \tag{4}$$

Similarly, we can define the concepts of drawup, and maximum drawup. *Drawup* U_t is defined as the increase of the asset price from its running minimum:

$$U_t = S_t - m_t, \tag{5}$$

where

$$m_t = \min_{u\in[0,t]} S_u. \tag{6}$$

Maximum drawup MDU_t is given by

$$MDU_t = \max_{u \in [0,t]} U_u. \tag{7}$$

Figure 1 illustrates the concepts of drawdown and drawup on data taken from S&P500 for the year 2005 with daily monitoring of the closing values. More frequent monitoring is also possible.

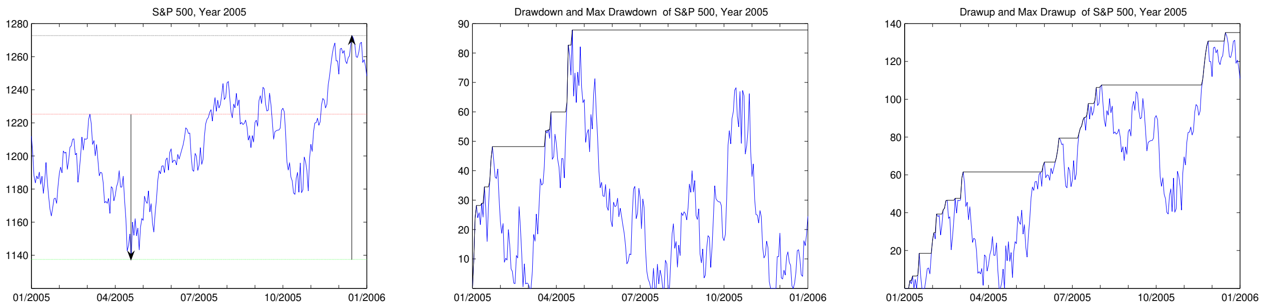

Figure 1: Left: S&P500 daily closing values in 2005. The downward arrow marks the maximum drawdown, the upward arrow marks the maximum drawup. Center: Drawdown (blue) and maximum drawdown (black) of S&P500 in 2005. Right: Drawup (blue) and maximum drawup (black) of S&P500 in 2005.

Maximum drawdown has been extensively studied in the recent literature. Risk measures based on the maximum drawdown can serve as an alternative to the commonly used Value-at-Risk. Portfolio optimization using the drawdown is considered in Chekhlov, Uryasev and Zabarkin (2005). Analytical results linking the maximum drawdown to the mean return appeared in the paper of Magdon-Ismail and Atiya (2004).

It is important to stress that the risk measures which depend on the maximum drawdown are directly observable at the end of the given period because they are functions of the data, and thus could be traded. In contrast, Value at Risk, or recently proposed Coherent Risk Measures (Artzner, Delbaen, Eber, Heath (1999)) do not give quantities which could be traded.

The price of a futures contract on maximum drawdown can serve as an important risk measure indicator which could be quoted by the market (rather than determined internally). When the market is in a bubble, it is reasonable to expect that the prices of drawdown contracts would be significantly higher. On the other hand, when the market is stable, or when it exhibits mean reversion behavior, the prices of drawdown contracts would become cheaper.

2 Comparison to Puts and Lookbacks

Currently traded contracts have only limited ability to insure the market drawdowns. For instance, deep out of the money puts are optimal to buy when the market reaches its maximum, otherwise they lose their value. This happens when the market goes up before it drops down. Thus timing is very important for put options. On the other hand, the maximum drawdown is not sensitive to timing, the contract delivers the largest drop no matter when this drop happens. The maximum drawdown does not lose value when the underlying asset is resetting its new maximum unlike put options.

Lookback options are contracts with the payoffs being either the drawdown D_T (lookback call), or the drawup U_T (lookback put). One disadvantage of these contracts is that the final drawdown (drawup) can end up far below the maximum drawdown (maximum drawup) in the given period. This can happen when the market drops down in the middle of the monitoring period, but recovers at the time of maturity of the contract. In our example of S&P 500 in year 2005, the drawdown ended up at 24.45, while the maximum drawdown was at 87.81.

Notice that the maximum drawdown is not replicated by a lookback call minus a lookback put option due to the requirement for the maximum to occur before the minimum in the drawdown payoff.

3 Pricing and Hedging

As for the pricing, the value $v(t, S_t, M_t, MDD_t)$ of any type of contract that depends on the maximum drawdown is given by taking the conditional expectation of the discounted payoff under the risk neutral measure:

$$v(t, S_t, M_t, MDD_t) = \mathbb{E}[e^{-r(T-t)} f(\{MDD_u\}_{u=0}^{T}) | S_t, M_t, MDD_t]. \quad (8)$$

A similar formula applies for the pricing of contracts that depend on the maximum drawup, MDU_t. Here, the function f determines the type of payoff defined by the contract (for instance, $f(\{MDD_t\}_{t=0}^{T}) = MDD_T - K$ for the forward contract, etc.). For the evolution of the underlying asset under the risk neutral measure, we may assume that

$$dS_t = rS_t dt + g(t, S_t) dN_t, \quad (9)$$

for a general martingale N_t (diffusion or jump type process).

The price can be computed by using Monte Carlo simulations. The pricing via conditional expectations can be also linked to partial differential equations (PDE) through the Feynman-Kac theorem (see for instance Shreve (2004)).

The hedging strategy is given by a standard delta hedge:

$$\Delta(t) = v_s(t, S_t, M_t, MDD_t). \quad (10)$$

This computation is also straightforward if one adopts Monte Carlo simulation (see Glasserman (2004)), or by using the finite difference method applied to the corresponding pricing PDE. Figure 2 shows the rescaled price of the forward contract on the maximum drawdown MDD_T as a function of the ratio of the drawdown and the stock and the difference of the MDD and the drawdown divided by the stock, assuming the geometric Brownian motion model. It also shows the hedging strategy, which is quite intuitive. When the stock is near its maximum (drawdown is close to zero), or when the current drawdown is far from the level of the maximum drawdown, the hedging position is close to zero (limited exposure to the market). However, when the current drawdown is close to the maximum drawdown, there is the potential that a new level of the maximum drawdown will be set, and thus the hedging position is to short some fraction of the index. One can prove that the hedge never gets below -1.

Notice that the hedging strategy has some of the same features of *momentum trading*. The hedger should be short when the market is setting new record lows. Similarly, the momentum trader assumes that the given trend will continue and thus takes also a short position in the falling market. Momentum traders essentially believe that the realized maximum drawdown (maximum drawup or range) will be larger than expected, and thus they are natural buyers of this contract. On the other hand, selling the (unhedged) contract is equivalent to taking the opposite strategy, namely buying the asset when it is setting its new low. This is known as the *contrarian trading*. Contrarian traders believe that the realized maximum drawdown (maximum drawup or range) will be smaller than expected, and they are natural sellers of this contract.

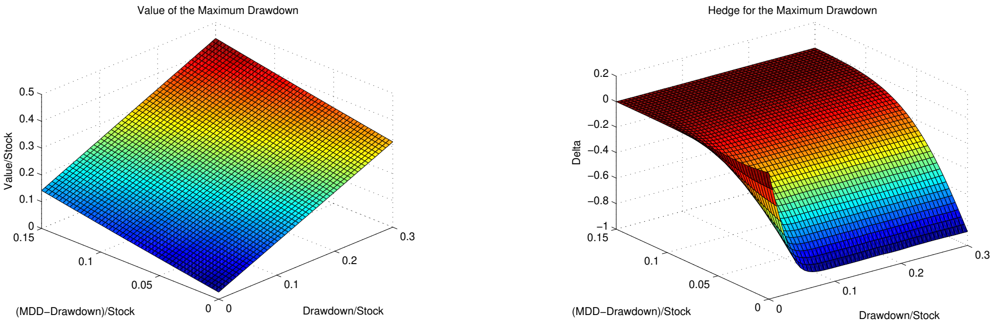

Figure 2: Left: The price of the contract with the payoff MDD_T (maximum drawdown) as a function of the ratio of the drawdown and the stock and the difference of the MDD and the drawdown divided by the stock, assuming the geometric Brownian motion model, using interest rate $r = 0.03$, volatility $\sigma = 11\%$, time to maturity $T = 1/4$. Right: The corresponding hedging strategy.

Table 1 lists realized values and theoretical prices of contracts depending on the maximum drawdown or drawup written on S&P500 in the year 2005. Notice that in that period, theoretical prices of 1 year contracts written on both drawdown and drawup exceeded the realized values. This might be due to the effect of mean reversion – stable real markets could stay within a more conservative range in the long run if compared to models based on geometric Brownian motion model. We assumed a rather small estimate of volatility $\sigma = 11\%$ implied from one year options on S&P 500 at that time. This period of time would have led to significant profits for the sellers of the MDD contract and the contrarian traders in general.

Contracts Expiring on 12/30/2005, S&P500	Realized Values		Theoretical Prices	
	3 months	1 year	3 months	1 year
Forward on MDD	51.97	87.81	71.14	141.25
Forward on MDU	95.90	135.24	78.80	171.19

Table 1: The realized values and theoretical prices of forward contracts on the maximum drawdown, and, the maximum drawup, where the underlying index is S&P500. We assume that the 3 month contract starts on Oct 3, 2005 and expires on Dec 30, 2005; the 1 year contract starts on Jan 3, 2005 and expires on Dec 30, 2005. We use opening value 1,211.92 of the index for the one year contract, opening value 1228.81 of the index for the 3 months contract, interest rate $r = 0.03$, volatility $\sigma = 11\%$. Theoretical prices are obtained from Monte Carlo simulation, standard deviations do not exceed 0.01.

Figure 3 shows the evolution of the price of the MDD contract (daily monitoring) in year 2005 and its corresponding hedging strategy. The green line is the theoretical price at each given time, converging to the realized MDD at the end. This could be regarded as a futures contract on MDD. The red line represents the value of the hedging portfolio which was constructed by daily rebalancing using the delta hedge computed by Monte Carlo simulation. We assumed a simple geometric Brownian motion model and a constant volatility of 11%. Notice that the value of the MDD contract was declining over that period of time, giving potentially substantial returns to the sellers (contrarian traders). The hedging portfolio replicates the MDD contract remarkably closely, suggesting that MDD is quite a robust statistic which is not affected too much by fluctuations of the parameters (such as the stochastic volatility). The MDD contract is increasing as a function of volatility as are most derivative contracts.

Figure 4 shows the opposite situation, namely the largest 3 months drop of S&P 500 in the history (May – July 2002; 308.89 points dropping from 1106.59 to 797.70). The market was also more volatile, with a volatility estimate of 25%. The hedging portfolio almost perfectly replicates this event as well. Notice that the price of the MDD contract more than doubled in 3 months, giving a remarkable return to the buyer (or the momentum trader). The hedging strategy stays close to a unit short position, realizing this significant return in the falling market.

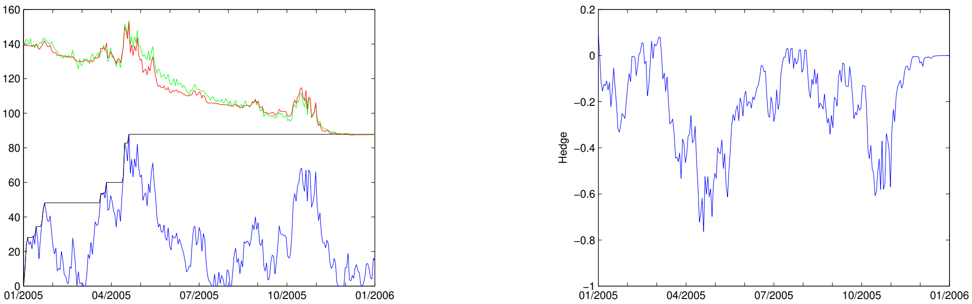

Figure 3: Left: Drawdown (blue), Maximum drawdown (black), price of the MDD contract (green) and the value of the hedging portfolio (red). The underlying asset is the index S&P500 in 2005 with opening value 1,211.92, interest rate $r = 0.03$, volatility $\sigma = 11\%$. Right: The corresponding hedge.

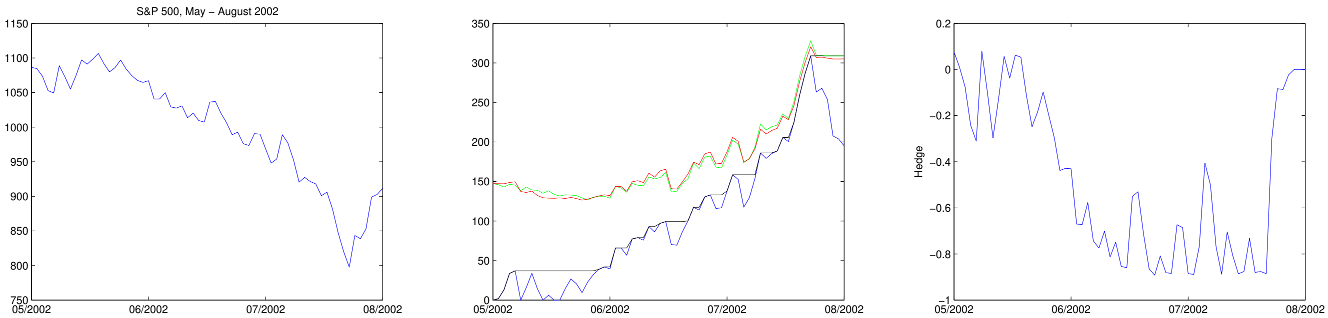

Figure 4: The worst 3 months drawdown in S&P 500 history (May – July 2002). Left: S&P 500 in the given period. Center: Drawdown (blue), Maximum drawdown (black), price of the MDD contract (green) and the value of the hedging portfolio (red). Opening value 1086.46, interest rate $r = 0.03$, volatility $\sigma = 25\%$. Right: The corresponding hedge.

4 MDD and MDU Calls, Puts and Barriers (Crash and Rally Options)

One natural step is to extend the universe of Maximum drawdown (MDD) or Maximum drawup (MDU) contracts to the usual derivative contracts, such as plain vanilla or barrier options. We may consider call or put options on the MDD (MDU) with payoffs

$$(MDD_T - K)^+, \quad \text{or} \quad (MDU_T - K)^+ \tag{11}$$

for the *call options*, and

$$(K - MDD_T)^+, \quad \text{or} \quad (K - MDU_T)^+ \tag{12}$$

for the *put options*. Figure 5 gives the value of the one year option on the MDD as a function of the strike.

Figure 6 gives the hedging strategy for the call option on the Maximum drawdown as a function of the ratio of the drawdown and the stock and the difference of the MDD and the drawdown divided by the stock, assuming the geometric Brownian motion model. Notice that the hedge for the call is very similar to hedging the maximum drawdown itself, but it is close to zero in the region when the drawdown is below the strike. Thus all the significant hedging happens once the MDD exceeds the strike. MDD is a nondecreasing process and when the call option gets in the money, it will expire in the money.

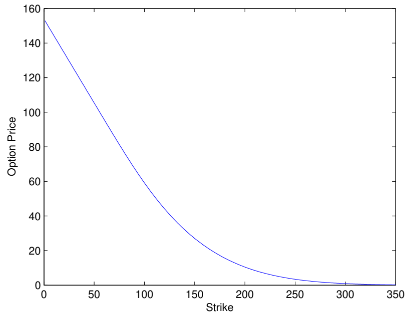

Figure 5: The price of the call option $(MDD_1 - K)^+$ as a function of the strike K, same underlying index S&P500, the value of the option on 01/03/2005 with maturity 12/30/2005.

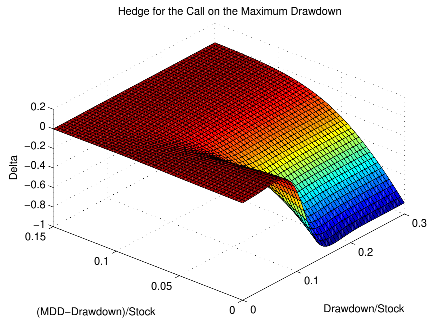

Figure 6: The hedge of the call option on MDD as a function of the ratio of the drawdown and the stock and the difference of the MDD and the drawdown divided by the stock, assuming the geometric Brownian motion model, using interest rate $r = 0.03$, volatility $\sigma = 11\%$, time to maturity $T = 1/4$, strike $K = 0.15S_0$.

Similarly, we can consider knock-in barrier option on MDD and MDU respectively. Let us define the event of the market crash as the first time the stock price S_t drops by a constant a from its running maximum, i.e.,

$$T_a = \inf\{t \geq 0 : D_t = M_t - S_t \geq a\} = \inf\{t \geq 0 : MDD_t \geq a\}, \tag{13}$$

which is obviously the same event as the first time the maximum drawdown exceeds level a.

A *crash option* is defined as a contract which pays off a at the time of the market crash T_a if $T_a < T$, where T is the maturity. Other payoffs, such as $M_{T_a} - S_{T_a}$ ($M_{T_a} - S_{T_a}$ could be greater than a if the market exhibits price discontinuity at the time of the crash), are possible to consider. The crash option can be regarded as a barrier option on the maximum drawdown.

The closest contract to the crash option that has been studied in the literature is the Russian option. It is a perpetual option that, at the time of the exercise, pays off the running maximum (discounted by a rate $\alpha > r$) of the asset price. It turns out that in the Black-Scholes model, the optimal exercise time is the time of a drawdown of a certain size a. See, for instance, Shepp and Shiryaev (1993).

The crash option resets its holder to the historical maximum of the asset price during the lifetime of the contract at the time of the market crash. If a crash does not happen up to the time of the maturity of the contract, the option expires worthless. Since the market crash is a rather extreme event, most of the time the option will not end up in the money if we are considering a large drop. This feature will make the contract cheap. It is possible to construct a contract which would set the wealth of its holder to a different level than the running maximum at the time of the crash, for instance to the running average, or to some intermediate point between the running maximum and the crash value. However, hedging of the barrier contracts might lead to huge delta positions when the option is about to expire, and the underlying process is close to knock in. This is a general feature of similar barrier contracts.

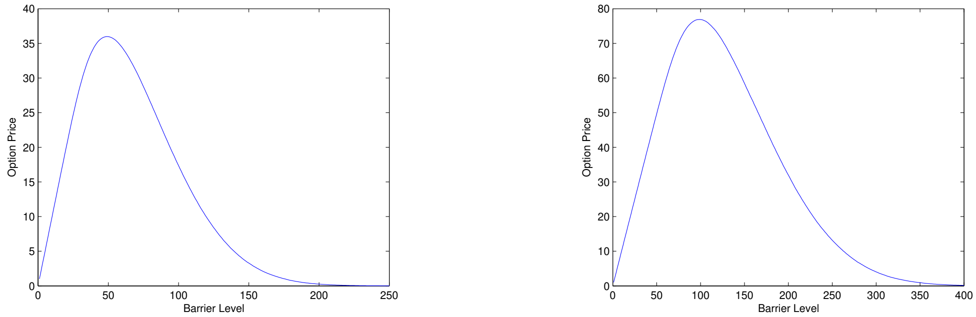

Figure 7: Price of the crash option (with daily monitoring) as a function of barrier level and maturities 3 months (left) and 1 year (right). We assume that the 3 month contract starts on Oct 3, 2005 and expires on Dec 30, 2005; the 1 year contract starts on Jan 3, 2005 and expires on Dec 30, 2005. We use opening value 1,211.92 of the index for the one year contract, opening value 1228.81 of the index for the 3 months contract, interest rate $r = 0.03$, volatility $\sigma = 11\%$.

Figure 7 plots the theoretical price of the crash option as a function of the barrier level. Notice that small drawdowns typically happen within the given time framework simply from the random nature of the price process, making the price of crash option increase almost linearly around zero. However, drops of larger sizes become increasingly unlikely, making the corresponding option price cheaper.

A *rally option* insures the opposite event of a crash, the case of a market rally. Define the rally event in the absolute value as the first time the stock price raises by a constant b from its running minimum, i.e.,

$$T_b = \inf\{t \geq 0 : S_t - m_t \geq b\} = \inf\{t \geq 0 : MDU_t \geq b\}. \tag{14}$$

The rally option is defined as a contract which pays off b at the time of the rally T_b if $T_b < T$, where T is the maturity. Other payoffs, such as $S_{T_b} - m_{T_b}$, are possible to consider.

All contracts described in this paper can be also defined using the percentage (relative value) drawdown or drawup as opposed to the absolute values. For instance, we can define the *maximum relative drawdown* $MRDD_t$ as the largest relative drop of the asset with respect to its running maximum up to time t:

$$MRDD_t = \max_{u \in [0,t]} (1 - \tfrac{S_u}{M_u}). \tag{15}$$

One can trade various options with maximum relative drawdown as the underlying process. For example, the barrier option on $MRDD_t$ represents the relative value crash option with the payoff $a^* \cdot M_{T_{a^*}}$ at the first time T_{a^*} when the relative drawdown exceeds $100 \cdot a^*$ percent:

$$T_{a^*} = \inf\{t \geq 0 : MRDD_t \geq a^*\}. \tag{16}$$

Similarly, we can define the *maximum relative drawup* as

$$MRDU_t = \max_{u \in [0,t]} \tfrac{S_u}{m_u} - 1, \tag{17}$$

and the corresponding contracts written on $MRDU_t$ as the underlying process.

5 Acknowledgement

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