Options on a Traded Account: Vacation Calls, Vacation Puts and Passport Options*

Steven E. Shreve, Jan Večeř

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213. Emails: shreve@andrew.cmu.edu, vecer@andrew.cmu.edu.

September 21, 2000

Abstract. In this article we study options on a traded account. In terms of the actions available to the buyer, the options we study are more general than a class of options known as *passport options*; in terms of the model of the underlying asset they are more restrictive. Using probabilistic techniques, we find the value of these options, the optimal strategy of the buyer, and the hedging strategy the seller should use in response to a (not necessarily optimal) strategy by the buyer.

Key words: Passport options, vacation options, stochastic control, Hamilton-Jacobi-Bellman equation, comparison theorem, put-call parity, hedging

JEL classification: G13

Mathematics Subject Classification (1991): 90A09, 60H30, 60G44

1 Introduction

An option on a traded account is a contract which allows the holder of the option to switch during the life of the option among various positions in an underlying asset. In this paper, the holder's trading is financed by investing or borrowing at the interest rate r. The holder accumulates gains and losses resulting from this trading, and at the expiration of the option he gets the call option payoff with strike 0 on his final account value, i.e., he keeps any gain from trading and is forgiven any loss.

Suppose that the underlying asset evolves according to the equation

$$dS_t = S_t(rdt + \sigma dW_t), \tag{1.1}$$

*This work was supported by the National Science Foundation under grants DMS-95-00626 and DMS-98-02464.

where r is the interest rate, σ is the volatility of the underlying asset and W_t is a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{F}_t$ the σ-algebra generated by the Brownian motion up to time t, augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. For the sake of simplicity, we shall refer to this asset as a *stock*; in fact, it is often an exchange rate or a futures price.

Denote the option holder's trading strategy by q_t, the number of shares held at time t. The strategy q_t must satisfy the contractual constraint $q_t \in [\alpha, \beta]$, where $\alpha \leq \beta$. It turns out that the holder of the option should never take an intermediate position, i.e, at any time he should hold either α shares of stock or β shares. When $\alpha = \beta = 1$, the option shall be seen to reduce to a European call; when $\alpha = \beta = -1$, it reduces to a European put.

When $\alpha = -1$ and $\beta = 1$, so that at any time the option holder should be either long or short one share of stock, we have *the passport option* studied by Hyer et. al. [7], Andersen et. al. [1], Henderson & Hobson [6], Delbaen & Yor [2] and Nagayama [9]. Hyer et. al. [7] consider a case in which there is a cost of carry for holding the underlying asset and/or trading in the asset is not necessarily financed at the interest rate. They produce a closed-form solution for the value of the passport option when the cost of carry is zero and financing is at the interest rate (their "symmetric case"). Their method for doing this is to construct the associated Hamilton-Jacobi-Bellman equation, change the variables so as to reduce the dimensionality of the problem, and solve the resulting equation via a Green's function. They show in the symmetric case that the option holder should short the stock when the value of his trading account is positive, and he should hold a long position in the stock when the value of his trading account is negative. Hyer et. al. [7] also provide numerical results for the more general setting, as does Nagayama [9]. Andersen et. al. [1] solve for the passport option holder's optimal trading strategy when the cost of financing trading is not part of the option account, and the asset is allowed to pay dividends or have a cost of carry. Their methodology is to reduce the dimensionality of the problem by changing the numéraire and then analyze the Hamilton-Jacobi-Bellman equation. When the dividend rate agrees with the interest rate, the option holder should use the "short when ahead, long when behind" strategy described above. Andersen et. al. [1] also provided numerical computation of the option value. Hobson & Henderson [6] show that in the symmetric case of Hyer et. al. the "short when ahead, long when behind" strategy remains optimal if the stock volatility is not constant, provided the volatility is an increasing function of the stock price. Their arguments are based on stochastic coupling.

Delbaen & Yor [2] and Hobson & Henderson [6] independently developed a clever argument based on local time and Skorohod's lemma to compute the value of the passport option in the symmetric case. Delbaen & Yor [2] extend this argument to obtain a new theorem about martingales, study the case with nonzero interest rate via the hitting time for a Bessel process, and solve the symmetric case when trading is permitted only at discrete times. They observe that there are many possible variations on the passport option, one of them being the constraint $\alpha \leq q_t \leq \beta$.

This variation is the subject of the present paper. The passport option is a special case, and we also single out the special cases we call the *vacation call option* corresponding to $0 \leq q_t \leq 1$ and the *vacation put option* corresponding to $-1 \leq q_t \leq 0$. These shall be seen to generalize the American call and the American put. We shall establish a "put-call parity" formula relating these options.

The passport option has some undesirable properties. It is expensive and if the holder wants to follow the optimal strategy, he must switch his position in the underlying asset every time his account value crosses zero. If he starts at zero, in the probabilistic model his account value will cross infinitely many times in an arbitrarily small interval of time. In reality one can switch only a finite number of times, but the theoretical result suggests that the holder of the option would have a good incentive to switch many times, thus imposing large operational costs. Moreover, the switches are between extremes; for the passport option, it is never optimal to have a zero position in the underlying asset.

These properties of the passport option suggest that one should consider contracts which would not cause the optimal holder to switch in the underlying asset so often. Vacation call options ($0 \leq q_t \leq 1$) and vacation put options ($-1 \leq q_t \leq 0$) satisfy this property. These contracts resemble the American call and American put, respectively. The vacation call and vacation put give more choices to the holder than the American call and American put, but for almost the same price. This is similar to the relationship between American and European options on non-dividend paying asset; the American option gives more optionality than the European option for the same or nearly the same price.

In contrast to previous papers, our arguments are entirely probabilistic. Using a Mean Comparison Theorem, we quickly determine the optimal trading strategy for the option holder. The computation of the price of the option is more complicated, but is executed using probabilistic arguments. We also discuss the hedging strategy which the option seller should use in response to the (not necessarily optimal) trading strategy employed by the option holder.

The paper is organized as follows. Section 2 formulates the model. Section 3 presents known results for passport options and develops the relationship between European, American and vacation calls and puts. For the general option on a traded account, the optimal strategy for the option holder is determined by Hajek's Mean Comparison Theorem in Section 4. Section 5 develops a generalization of put-call parity for these options. In Section 6 we perform the tedious but straight-forward computation of the option value, and Section 7 specializes this to the vacation call and put. Section 8 considers the hedging problem for the option seller, and Section 9 summarizes the paper.

2 The Model Formulation

In our model the value of the option holder's account corresponding to the strategy q satisfies

$$\begin{aligned} dX_t^q &= q_t dS_t + r(X_t^q - q_t S_t)dt \;=\; rX_t^q dt + q_t \sigma S_t dW_t, \qquad (2.1)\\ X_0^q &= X_0. \end{aligned}$$

This represents a self-financing trading strategy in the money market and the underlying asset with the initial wealth X_0. The holder of the option will receive at time T the payoff $[X_T^q]^+$. The objective of the seller of the option, who makes this payment, is to be prepared to hedge against all possible strategies of the holder of the option. As we shall see, this means the price of this contract at time t should be the maximum over all possible strategies q_u of the discounted expected value under the risk-neutral probability $\mathbb{P}$ of the payoff of the option, i.e.,

$$V^{[\alpha,\beta]}(t, S_t, X_t) = \max_{q_u \in [\alpha,\beta]} e^{-r(T-t)} \mathbb{E}[[X_T^q]^+ | \mathcal{F}_t], \quad t \in [0, T]. \tag{2.2}$$

Computation of the expression in (2.2) is a problem of stochastic optimal control, and the function $V^{[\alpha,\beta]}(t, s, x)$ is characterized by the corresponding Hamilton–Jacobi–Bellman (HJB) equation

$$-rV + V_t + rsV_s + rxV_x + \max_{q \in [\alpha,\beta]} \tfrac{1}{2}\sigma^2 s^2 (V_{ss} + 2qV_{sx} + q^2 V_{xx}) = 0 \tag{2.3}$$

with the boundary condition

$$V(T, s, x) = x^+. \tag{2.4}$$

The maximum in (2.3) is attained by the optimal strategy q_t^{opt}. Except for the constraint imposed on q_t, this equation is the same for a wide range of options on a traded account. One way to determine the price of an option on a traded account is to directly attack (2.3), (2.4). The normal method of using the HJB equation is to first find a function which satisfies it, and then verify that this is the value of the option. As mentioned earlier, we follow a probabilistic route. We first determine the value of the option $V^{[\alpha,\beta]}$, and then we can verify it satisfies (2.3), (2.4).

3 Passport and Vacation Options

In the event that $[\alpha, \beta] = [-1, 1]$, we have the passport option discussed earlier. The value of this option, determined by the papers cited earlier and rederived later in this paper, is

$$\begin{aligned} PP(t, S_t, X_t) &\triangleq V^{[-1,1]}(t, S_t, X_t) \qquad (3.1)\\ &= \tfrac{1}{2}\Big(X_t + (S_t + X_t)N(d_+) - S_t N(d_-) - S_t\bar{\sigma} d_- N(-d_-) + \tfrac{S_t\bar{\sigma}}{\sqrt{2\pi}} \exp\Big(-\tfrac{d_-^2}{2}\Big)\Big) \end{aligned}$$

for $X_t \geq 0$, where

$$d_\pm = \tfrac{1}{\bar{\sigma}} \log(1 + \tfrac{X_t}{S_t}) \pm \tfrac{\bar{\sigma}}{2}, \quad \bar{\sigma} = \sigma\sqrt{T - t},$$

and

$$\begin{aligned} &PP(t, S_t, X_t) \qquad (3.2)\\ &= \tfrac{1}{2}\Big(X_t + (S_t - X_t)N(\epsilon_+) - S_t N(\epsilon_-) - S_t\bar{\sigma}\epsilon_- N(-\epsilon_-) + \tfrac{S_t\bar{\sigma}}{\sqrt{2\pi}} \exp\Big(-\tfrac{\epsilon_-^2}{2}\Big)\Big) \end{aligned}$$

for $X_t \le 0$, where

$$\epsilon_\pm = \tfrac{1}{\bar\sigma}\log(1 - \tfrac{X_t}{S_t}) \pm \tfrac{\bar\sigma}{2}, \quad \bar\sigma = \sigma\sqrt{T-t}.$$

Here $N(x)$ denotes the standard normal cumulative distribution function. The optimal strategy for the passport option is

$$q_t^{opt} = -\mathrm{sgn}\left(X_t^{q^{opt}}\right), \tag{3.3}$$

i.e., be short when ahead and be long when behind. We adopt the convention that

$$\mathrm{sgn}(0) = 1. \tag{3.4}$$

In this paper, we introduce the *vacation call* ($\alpha = 0$, $\beta = 1$) and the *vacation put* ($\alpha = -1$, $\beta = 0$) as cheaper alternatives to the passport option. These resemble the American call and American put, respectively. To see these connections, we note that a European call gives the holder the payoff $(S_T - K)^+$, where K is the strike. In terms of an option on a traded account, one may write

$$\begin{aligned} dX_t &= rX_t dt + q_t \sigma S_t dW_t, \quad q_t \equiv 1, && (3.5)\\ X_0 &= S_0 - e^{-rT}K, && (3.6) \end{aligned}$$

which represents the value of the traded account provided the option holder is always long one share of the stock. The solution of (3.5), (3.6) is

$$X_t = S_t - e^{-r(T-t)}K, \tag{3.7}$$

and therefore, $X_T = S_T - K$. The European call is an option on a traded account with the constraint $q_t \equiv 1$.

Suppose now we give the owner of an option on a traded account the right to switch from $q_t = 1$ to $q_t = 0$ exactly once during the life of the option. If he switches at time t, he will at that time receive $X_t = S_t - e^{-r(T-t)}K$, which he then invests in the money market until time T, so that $X_T = e^{r(T-t)}S_t - K$. This is similar to an American call, exercised at time t, except the strike price is discounted for the time between exercise and expiration. In this light, the vacation call is a natural extension of the American call. Switching to a zero position in the vacation call contract does not terminate it; unlike the case of the American call, the holder of the vacation call has the right to switch back to a long position. Figure 1 illustrates the relationship among the value EC of a European call, the value VC of a vacation call, and the value PP of a passport option as functions of volatility of the underlying stock:

$$EC \triangleq V^{[1,1]} \le VC \triangleq V^{[0,1]} \le PP \triangleq V^{[-1,1]}.$$

In this figure, $X_0 = 0$ and $r = 0$, which means that the call is struck at-the-money. Notice that for all reasonable volatilities the prices of the vacation call and European call are indistinguishable. The passport option value increases rapidly with increasing volatility, and the holder of this option must switch positions often to realize this extra value.

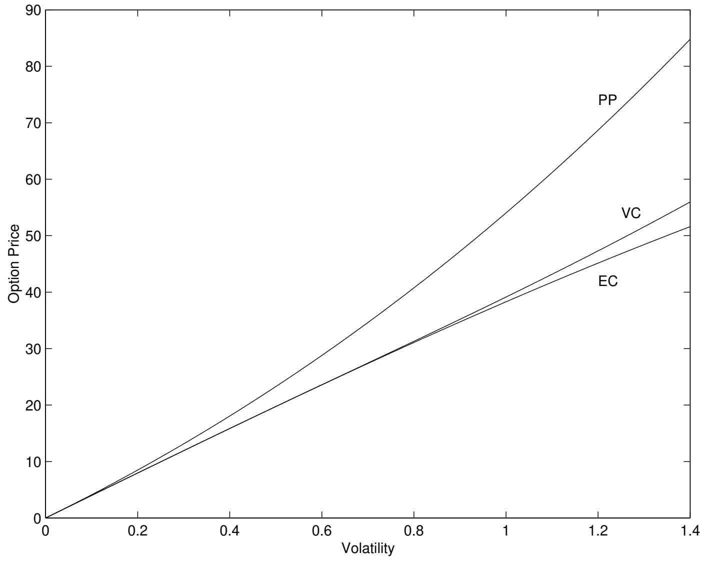

Figure 1: Prices of European Call (EC), Vacation Call (VC) and Passport Option (PP) as a function of volatility, $T = 1$, $S_0 = 100$, $X_0 = 0$.

We have an analogous situation with European put, which gives the holder the payoff $(K - S_T)^+$, where K is the strike. This is the same as an option on a traded account with the constraint of being always short in the underlying asset and with the initial wealth $X_0 = e^{-rT}K - S_0$. In other words,

$$dX_t = rX_t dt + q_t \sigma S_t dW_t, \quad q_t \equiv -1, \tag{3.8}$$

$$X_0 = e^{-rT}K - S_0, \tag{3.9}$$

has the solution

$$X_t = e^{-r(T-t)}K - S_t, \tag{3.10}$$

and in particular $X_T = K - S_T$.

More generally, if we have $q_t \equiv \alpha$, a constant position in the underlying asset, i.e.,

$$dX_t = rX_t dt + q_t \sigma S_t dW_t, \quad q_t \equiv \alpha,$$

then when $\alpha > 0$, the corresponding option price is (we write $V^{[\alpha]}$ rather than $V^{[\alpha,\alpha]}$)

$$V^{[\alpha]}(t, S_t, X_t) = \alpha S_t N(d_+) - (\alpha S_t - X_t) N(d_-),$$

for $X_t < \alpha S_t$, where

$$d_\pm = -\tfrac{1}{\bar\sigma} \log\left(1 - \tfrac{X_t}{\alpha S_t}\right) \pm \tfrac{\bar\sigma}{2}, \quad \bar\sigma = \sigma\sqrt{T-t}, \tag{3.11}$$

and

$$V^{[\alpha]}(t,S_t,X_t) = X_t$$

for $X_t \geq \alpha S_t$. When $\alpha < 0$, we have

$$V^{[\alpha]}(t,S_t,X_t) = \alpha S_t N(-d_+) - (\alpha S_t - X_t)N(-d_-),$$

for $X_t \geq \alpha S_t$, where again $d_\pm$ is given by (3.11). and

$$V^{[\alpha]}(t,S_t,X_t) = 0$$

for $X_t < \alpha S_t$. When $\alpha = 0$ we get trivially

$$V^{[0]}(t,S_t,X_t) = X_t^+.$$

If we set $\alpha = 1$, we get the price of the European call with strike $K_c = e^{r(T-t)}(S_t - X_t)$ (see (3.7))

$$EC(t,S_t,K_c) = V^{[1]}(t,S_t,X_t) \tag{3.12}$$

and for $\alpha = -1$ we get the price of the European put with strike $K_p = e^{r(T-t)}(S_t + X_t)$ (see (3.10))

$$EP(t,S_t,K_p) = V^{[-1]}(t,S_t,X_t). \tag{3.13}$$

4 The Optimal Strategy

We now return to the general constraint $\alpha \leq q_t \leq \beta$, $0 \leq t \leq T$, and in this section determine the optimal strategy for the holder of the option on a traded account. Using the change of variable

$$Z_t^q = \tfrac{X_t^q}{S_t} - \tfrac{\alpha+\beta}{2}, \tag{4.1}$$

we can reduce the dimensionality of the problem, as we show below. Applying Itô's formula to the process Z_t^q, we get

$$dZ_t^q = \left(Z_t^q + \tfrac{\alpha+\beta}{2} - q_t\right)\sigma^2 dt - \left(Z_t^q + \tfrac{\alpha+\beta}{2} - q_t\right)\sigma dW_t.$$

We next define a new probability measure $\widetilde{\mathbb{P}}$ by

$$\widetilde{\mathbb{P}}(A) = \int_A D_T d\mathbb{P}, \quad A \in \mathcal{F},$$

where

$$D_T = e^{-rT} \cdot \tfrac{S_T}{S_0} = \exp\left(\sigma W_T - \tfrac{1}{2}\sigma^2 T\right).$$

Under $\widetilde{\mathbb{P}}$, $\widetilde{W}_t = -\sigma t + W_t$ is a Brownian motion, according to Girsanov's theorem. Notice that

$$e^{-rT}\mathbb{E}[X_T^q]^+ = e^{-rT}\widetilde{\mathbb{E}}\left[\frac{X_T^q}{D_T}\right]^+ = S_0 \cdot \widetilde{\mathbb{E}}\left[\frac{X_T^q}{S_T}\right]^+ = S_0 \cdot \widetilde{\mathbb{E}}\left[Z_T^q + \frac{\alpha+\beta}{2}\right]^+$$

and

$$dZ_t^q = -\left(Z_t^q + \tfrac{\alpha+\beta}{2} - q_t\right)\sigma d\widetilde{W}_t. \tag{4.2}$$

Thus our problem reduces to finding q_t^{opt} which maximizes

$$\max_{q_t \in [\alpha,\beta]} \widetilde{\mathbb{E}}\left[Z_T^q + \tfrac{\alpha+\beta}{2}\right]^+ \tag{4.3}$$

and computing the corresponding option value

$$V^{[\alpha,\beta]}(t, S_t, X_t^{q^{opt}}) = S_t \cdot u^{[\alpha,\beta]}(t, Z_t^{q^{opt}}), \tag{4.4}$$

where

$$u^{[\alpha,\beta]}(t,z) \triangleq \widetilde{\mathbb{E}}\left[\left(Z_T^{q^{opt}} + \tfrac{\alpha+\beta}{2}\right)^+ \Big| Z_t = z\right]. \tag{4.5}$$

Using a comparison theorem of Hajek [5] (restated as Theorem 4.2 below), we will show that (4.3) is maximized by maximizing the absolute value of the volatility term $-\left(Z_t + \frac{\alpha+\beta}{2} - q_t\right)\sigma$ for all t and ω. In other words

$$q_t^{opt} = \alpha I_{\{(Z_t^{q^{opt}}) \geq 0\}} + \beta I_{\{(Z_t^{q^{opt}}) < 0\}}, \tag{4.6}$$

where I denotes the indicator function. Before we prove this result, we need to know that the stochastic differential equation obtained by substituting q_t^{opt} into (4.2) has a unique solution.

Theorem 4.1 *The stochastic differential equation*

$$\begin{aligned} dZ_t &= -\left(Z_t + \tfrac{\beta-\alpha}{2} \cdot \operatorname{sgn}(Z_t)\right)\sigma d\widetilde{W}_t \\ Z_0 &= z, \end{aligned} \tag{4.7}$$

where convention (3.4) is used, has a unique solution in the sense of probability law.

Proof. Both existence and uniqueness follow from [3] (or see Theorem 5.5.7 in Karatzas and Shreve [8], page 335). ◇

Theorem 4.2 (Hajek's Mean Comparison Theorem) *Let M_t be a continuous martingale with representation $M_t = M_0 + \int_0^t \sigma_s dW_s$. Assume that for some function ρ on $\mathbb{R}$, we have $|\sigma_s| \leq \rho(M_s)$ and there exists a unique (in the sense of probability law) solution N_t to the stochastic differential equation*

$$N_t = M_0 + \int_0^t \rho(N_s) dW_s. \tag{4.8}$$

Then for any convex function Φ and any $t \geq 0$,

$$\mathbb{E}\Phi(M_t) \leq \mathbb{E}\Phi(N_t).$$

PROOF. This is Theorem 3 in Hajek [5] with the exception that the theorem in [5] requires Lipschitz continuity of the function ρ. However, all that is required by the proof of Hajek's theorem is that the stochastic differential equation (4.8) has a solution which is unique in the sense of probability law. $\diamond$

Notice that

$$\tfrac{\alpha+\beta}{2} - (\alpha I_{\{x\geq 0\}} + \beta I_{\{x<0\}}) = \tfrac{\beta-\alpha}{2} \cdot \operatorname{sgn}(x).$$

Theorems 4.1 and 4.2 applied to (4.2) and the function

$$\rho(x) = -\left(x + \tfrac{\beta-\alpha}{2} \cdot \operatorname{sgn}(x)\right)\sigma$$

lead to the following conclusion.

Theorem 4.3 (The Optimal Strategy) *The optimal strategy for the holder of the option on the traded account is*

$$q_t^{opt} = \alpha I_{\{(Z_t^{q^{opt}})\geq 0\}} + \beta I_{\{(Z_t^{q^{opt}})<0\}}. \tag{4.9}$$

Thus the optimal strategy as a function of the money account and the stock price is

$$q_t^{opt} = \alpha I_{\{(X_t^{q^{opt}})\geq \frac{\alpha+\beta}{2} S_t\}} + \beta I_{\{(X_t^{q^{opt}})< \frac{\alpha+\beta}{2} S_t\}}. \tag{4.10}$$

Remark 4.4 (Hamilton-Jacobi-Bellman equation reduction) If we apply the change of variable (4.1) to the HJB equation (2.3), (2.4) and introduce a function u by

$$V^{[\alpha,\beta]}(t,s,x) = s \cdot u^{[\alpha,\beta]}(t,z), \quad z = \tfrac{x}{s} - \tfrac{\alpha+\beta}{2},$$

we get

$$u_t + \tfrac{\sigma^2}{2} \max_{q\in[\alpha,\beta]} \left(\left(z + \tfrac{\alpha+\beta}{2} - q\right)^2 u_{zz}\right) = 0 \tag{4.11}$$

with the boundary condition

$$u^{[\alpha,\beta]}(T,z) = \left(z + \tfrac{\alpha+\beta}{2}\right)^+.$$

It turns out that $u_{zz} > 0$, and thus the maximum in (4.11) is attained by the optimal strategy (4.9). Moreover, the function u satifies the partial differential equation

$$\begin{aligned} u_t + \tfrac{\sigma^2}{2}\left(|z| + \tfrac{\beta-\alpha}{2}\right)^2 u_{zz} &= 0, \\ u^{[\alpha,\beta]}(T,z) &= \left(z + \tfrac{\alpha+\beta}{2}\right)^+. \end{aligned}$$

This change of variable is used by [7].

5 Generalized Put-Call Parity

There is a simple relationship between options on a traded account which generalizes the usual put-call parity.

Theorem 5.1 (Generalized Put-Call Parity) *Options on the traded account satisfy the following equation:*

$$V^{[\alpha,\beta]}(t,S_t,X_t) - V^{[-\beta,-\alpha]}(t,S_t,-X_t) = X_t.$$

PROOF. The change of variables (4.1) corresponding to the contract with the constraint $q_t \in [\alpha,\beta]$ is

$$Z_t^{[\alpha,\beta]} = \tfrac{X_t}{S_t} - \tfrac{\alpha+\beta}{2};$$

the change of variables corresponding to the contract with the constraint $q_t \in [-\beta,-\alpha]$ is

$$Z_t^{[-\beta,-\alpha]} = \tfrac{X_t}{S_t} + \tfrac{\alpha+\beta}{2}.$$

Both $Z_t^{[\alpha,\beta]}$ and $Z_t^{[-\beta,-\alpha]}$ with their respective optimal controls follow the same equation (see (4.7))

$$dZ_v = -\left(Z_v + \tfrac{\beta-\alpha}{2}\cdot \operatorname{sgn}(Z_v)\right)\sigma d\widetilde{W}_v.$$

According to (4.4), (4.5),

$$V^{[\alpha,\beta]}(t,S_t,X_t) = S_t \cdot \widetilde{\mathbb{E}}\left[\left(Z_T^{[\alpha,\beta]} + \tfrac{\alpha+\beta}{2}\right)^+ \middle| Z_t^{[\alpha,\beta]} = \tfrac{X_t}{S_t} - \tfrac{\alpha+\beta}{2}\right] \tag{5.1}$$

and

$$V^{[-\beta,-\alpha]}(t,S_t,X_t) = S_t \cdot \widetilde{\mathbb{E}}\left[\left(Z_T^{[-\beta,-\alpha]} - \tfrac{\alpha+\beta}{2}\right)^+ \middle| Z_t^{[-\beta,-\alpha]} = \tfrac{X_t}{S_t} + \tfrac{\alpha+\beta}{2}\right]. \tag{5.2}$$

In the problem with the constraint in $[\alpha,\beta]$, take the initial condition $S_t = s$, $X_t = x$. In the problem with the constraint in $[-\beta,-\alpha]$, take the initial condition $S_t = s$, $X_t = -x$. Then

$$\begin{aligned} -Z_t^{[-\beta,-\alpha]} &= \tfrac{x}{s} - \tfrac{\alpha+\beta}{2} = Z_t^{[\alpha,\beta]} \\ d\left(-Z_v^{[-\beta,-\alpha]}\right) &= -\left(\left(-Z_v^{[-\beta,-\alpha]}\right) + \tfrac{\beta-\alpha}{2}\cdot \operatorname{sgn}\left(-Z_v^{[-\beta,-\alpha]}\right)\right)\sigma d\widetilde{W}_v, \end{aligned}$$

and Theorem 4.1 shows that $-Z^{[-\beta,-\alpha]}(\cdot)$ and $Z^{[\alpha,\beta]}(\cdot)$ have the same distribution. Moreover, (5.2) can be rewritten as

$$\begin{aligned} &V^{[-\beta,-\alpha]}(t,S_t,-X_t) \\ &= S_t \cdot \widetilde{\mathbb{E}}\left[\left(Z_T^{[-\beta,-\alpha]} - \tfrac{\alpha+\beta}{2}\right)^+ \middle| Z_t^{[-\beta,-\alpha]} = -\tfrac{X_t}{S_t} + \tfrac{\alpha+\beta}{2}\right] \\ &= S_t \cdot \widetilde{\mathbb{E}}\left[\left(-Z_T^{[\alpha,\beta]} - \tfrac{\alpha+\beta}{2}\right)^+ \middle| Z_t^{[\alpha,\beta]} = \tfrac{X_t}{S_t} - \tfrac{\alpha+\beta}{2}\right] \\ &= S_t \cdot \widetilde{\mathbb{E}}\left[\left(Z_T^{[\alpha,\beta]} + \tfrac{\alpha+\beta}{2}\right)^- \middle| Z_t^{[\alpha,\beta]} = \tfrac{X_t}{S_t} - \tfrac{\alpha+\beta}{2}\right]. \end{aligned} \tag{5.3}$$

From the relationship $x = x^+ - x^-$, equations (5.1), (5.3), and the fact that $Z^{[\alpha,\beta]}$ is a martingale, we conclude

$$V^{[\alpha,\beta]}(t, S_t, X_t) - V^{[-\beta,-\alpha]}(t, S_t, -X_t)$$
$$= S_t \cdot \widetilde{\mathbb{E}}\left[Z_T^{[\alpha,\beta]} + \tfrac{\alpha+\beta}{2} \middle| Z_t^{[\alpha,\beta]} = \tfrac{X_t}{S_t} - \tfrac{\alpha+\beta}{2}\right] = X_t.$$

$\diamond$

Corollary 5.2 (European Put-Call Parity) We have in particular

$$V^{[1]}(t, S_t, X_t) - V^{[-1]}(t, S_t, -X_t) = X_t,$$

which is the same as (see (3.12), (3.13))

$$EC(t, S_t, K) - EP(t, S_t, K) = S_t - e^{-r(T-t)}K,$$

where

$$X_t = S_t - e^{-r(T-t)}K.$$

This is the usual put-call parity relationship.

Corollary 5.3 (Vacation Put-Call Parity) The vacation call has the constraint $q_t \in [0, 1]$, and the vacation put has the constraint $q_t \in [-1, 0]$. Put-call parity implies

$$V^{[0,1]}(t, S_t, X_t) - V^{[-1,0]}(t, S_t, -X_t) = X_t,$$

or in other words

$$VC(t, S_t, X_t) - VP(t, S_t, -X_t) = X_t.$$

Corollary 5.4 (Passport Put-Call Parity) The passport option has the constraint $q_t \in [-1, 1]$. Put-call parity implies

$$V^{[-1,1]}(t, S_t, X_t) - V^{[-1,1]}(t, S_t, -X_t) = X_t,$$

or

$$PP(t, S_t, X_t) - PP(t, S_t, -X_t) = X_t.$$

This relationship permits (3.2) to be easily derived from (3.1).

6 The Value of the Option on a Traded Account

It remains to compute the value of the option on a traded account. In this section we omit the superscript q^{opt} on the process $Z_t^{q^{opt}}$. We need to find

$$u^{[\alpha,\beta]}(t, z) = \widetilde{\mathbb{E}}\left[\left(Z_T + \frac{\alpha+\beta}{2}\right)^+ \middle| Z_t = z\right],$$

where Z_t follows the equation

$$\begin{aligned} dZ_v &= -\left(Z_v + \frac{\beta-\alpha}{2}\cdot \operatorname{sgn}(Z_v)\right)\sigma d\widetilde{W}_v, \\ Z_t &= z. \end{aligned}$$

We have then

$$V^{[\alpha,\beta]}(t,s,x) = s\cdot u^{[\alpha,\beta]}\left(t, \frac{x}{s} - \frac{\alpha+\beta}{2}\right),$$

according to (4.4). We have already derived the value of the option when $\alpha = \beta$. Here we treat the case $\alpha < \beta$.

Theorem 6.1 (Value of the option on a traded account) *Assume $\alpha < \beta$. Let $0 \le t < T$ be given, and set $\overline{\sigma} = \sigma\sqrt{T-t}$. When $0 \le \alpha < \beta$ or when $\alpha \le 0 \le -\alpha \le \beta$, the value of the option on a traded account is*

$$\begin{aligned} V^{[\alpha,\beta]}(t,s,x) &= \frac{(\alpha+\beta)^2 s}{4(\beta-\alpha)} - \frac{\alpha x}{\beta-\alpha} - \frac{1}{4}(\beta-\alpha)\overline{\sigma} s d_- \\ &\quad + \frac{1}{4\sqrt{2\pi}}(\beta-\alpha)\overline{\sigma} s \exp\left(-\tfrac{1}{2}d_-^2\right) + \frac{\beta}{\beta-\alpha}(x-\alpha s)N(d_+) \\ &\quad - \frac{1}{4}(\beta-\alpha)(1-\overline{\sigma} d_-) s N(d_-) \end{aligned} \tag{6.1}$$

for $x \ge \frac{1}{2}(\alpha+\beta)s$, where

$$d_\pm = \frac{1}{\overline{\sigma}}\log\left(\frac{2}{\beta-\alpha}\left(\frac{x}{s}-\alpha\right)\right) + \frac{1}{\overline{\sigma}}\log\left(\frac{2\beta}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2},$$

and

$$\begin{aligned} V^{[\alpha,\beta]}(t,s,x) &= x + \frac{1}{4\sqrt{2\pi}}(\beta-\alpha)\overline{\sigma} s \exp\left(-\tfrac{1}{2}e_{+-}^2\right) \\ &\quad - \frac{\beta}{\beta-\alpha}(x-\beta s)N(-e_{++}) - \frac{1}{4}(\beta-\alpha)(\overline{\sigma} e_{+-}+1) s N(-e_{+-}) \\ &\quad -(x-\beta s)N(e_{-+}) - \beta s N(e_{--}) \end{aligned} \tag{6.2}$$

for $x \le \frac{1}{2}(\alpha+\beta)s$, where

$$e_{\pm\pm} = \frac{1}{\overline{\sigma}}\log\left(\frac{2}{\beta-\alpha}\left(\beta-\frac{x}{s}\right)\right) \pm \frac{1}{\overline{\sigma}}\log\left(\frac{2\beta}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2}.$$

When $\alpha < \beta \le 0$ or when $\alpha \le 0 \le \beta \le -\alpha$, the value of the option on the traded account is

$$\begin{aligned} V^{[\alpha,\beta]}(t,s,x) &= \frac{1}{4\sqrt{2\pi}}(\beta-\alpha)\overline{\sigma} s \exp\left(-\tfrac{1}{2}d_{+-}^2\right) - \frac{\alpha}{\beta-\alpha}(x-\alpha s)N(-d_{++}) \\ &\quad - \frac{1}{4}(\beta-\alpha)(\overline{\sigma} d_{+-}+1) s N(-d_{+-}) \\ &\quad +(x-\alpha s)N(d_{-+}) + \alpha s N(d_{--}) \end{aligned} \tag{6.3}$$

for $x \geq \frac{1}{2}(\alpha+\beta)s$, *where*

$$d_{\pm\pm} = \frac{1}{\overline{\sigma}} \log\left(\frac{2}{\beta-\alpha}\left(\frac{x}{s}-\alpha\right)\right) \pm \frac{1}{\overline{\sigma}} \log\left(-\frac{2\alpha}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2},$$

and

$$\begin{aligned} V^{[\alpha,\beta]}(t,s,x) &= \frac{(\alpha+\beta)^2 s}{4(\beta-\alpha)} - \frac{\alpha x}{\beta-\alpha} - \frac{1}{4}(\beta-\alpha)\overline{\sigma} s e_- \\ &\quad + \frac{1}{4\sqrt{2\pi}}(\beta-\alpha)\overline{\sigma} s \exp\left(-\tfrac{1}{2}e_-^2\right) + \frac{\alpha}{\beta-\alpha}(x-\beta s)N(e_+) \\ &\quad - \frac{1}{4}(\beta-\alpha)(1-\overline{\sigma} e_-) s N(e_-) \end{aligned} \tag{6.4}$$

for $x \leq \frac{1}{2}(\alpha+\beta)s$, *where*

$$e_{\pm} = \frac{1}{\overline{\sigma}} \log\left(\frac{2}{\beta-\alpha}\left(\beta-\frac{x}{s}\right)\right) + \frac{1}{\overline{\sigma}} \log\left(-\frac{2\alpha}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2}.$$

Remark 6.2 It is straight-forward but extremely tedious to verify that the function $V^{[\alpha,\beta]}$ given above satisfies the HJB equation (2.3), with the maximizing q in that equation given by (cf. (4.10))

$$q^{opt} = \alpha I_{\{x \geq \frac{\alpha+\beta}{2}s\}} + \beta I_{\{x < \frac{\alpha+\beta}{2}s\}}.$$

We leave this to the reader.

Remark 6.3 (Passport option) When $\alpha = -1, \beta = 1$, formulas (6.1), (6.3) reduce to (3.1), (3.2), and (4.10) becomes (3.3).

Proof of Theorem 6.1. Define the strictly increasing, continuously differentiable, piecewise twice continuously differentiable function

$$y = f(z) = \frac{1}{\sigma} \cdot \operatorname{sgn}(z) \cdot \log\left(\frac{2}{\beta-\alpha}|z|+1\right),$$

so

$$|z| = \frac{\beta-\alpha}{2}\left(e^{\sigma|y|}-1\right).$$

Set $Y_t = f(Z_t)$. According to Itô's rule,

$$dY_t = -\operatorname{sgn}(Y_t)d\widetilde{W}_t - \frac{1}{2}\sigma \operatorname{sgn}(Y_t)dt.$$

From (5.14) on page 441 of Karatzas & Shreve [8], we have the transition density

$$\begin{aligned} &\widetilde{\mathbb{P}}(Y_{t+\tau} \in dx | Y_t = y) \\ &= \begin{cases} \frac{1}{\sqrt{2\pi\tau}}\left(e^{-\frac{1}{2\tau}\left(y-x-\frac{1}{2}\sigma\tau\right)^2} + \frac{1}{2}\sigma e^{-\sigma x}\int_{y+x}^{\infty} e^{-\frac{1}{2\tau}\left(v-\frac{1}{2}\sigma\tau\right)^2} dv\right) dx, \\ \qquad\qquad\qquad\qquad\qquad\qquad y \geq 0, x \geq 0, \\ \frac{1}{\sqrt{2\pi\tau}}\left(e^{\sigma y-\frac{1}{2\tau}\left(y-x+\frac{1}{2}\sigma\tau\right)^2} + \frac{1}{2}\sigma e^{\sigma x}\int_{y-x}^{\infty} e^{-\frac{1}{2\tau}\left(v-\frac{1}{2}\sigma\tau\right)^2} dv\right) dx, \\ \qquad\qquad\qquad\qquad\qquad\qquad y \geq 0, x \leq 0. \end{cases} \end{aligned}$$

Now

$$Z_T + \frac{\alpha+\beta}{2} = f^{-1}(Y_T) + \frac{\alpha+\beta}{2} = \begin{cases} \frac{\beta+\alpha}{2}e^{\sigma Y_T} + \alpha, & Y_T \geq 0, \\ \beta - \frac{\beta-\alpha}{2}e^{-\sigma Y_T}, & Y_T \leq 0. \end{cases}$$

When $0 \leq \alpha < \beta$ or when $\alpha \leq 0 \leq -\alpha \leq \beta$,

$$\left(Z_T + \frac{\alpha+\beta}{2}\right)^+ = \begin{cases} \frac{\beta-\alpha}{2}e^{\sigma Y_T} + \alpha, & Y_T \geq 0, \\ \beta - \frac{\beta-\alpha}{2}e^{-\sigma Y_T}, & -\frac{1}{\sigma}\log\left(\frac{2\beta}{\beta-\alpha}\right) \leq Y_T \leq 0, \\ 0, & Y_T \leq -\frac{1}{\sigma}\log\left(\frac{2\beta}{\beta-\alpha}\right). \end{cases}$$

When $\alpha < \beta \leq 0$ or when $\alpha \leq 0 \leq \beta \leq -\alpha$,

$$\left(Z_T + \frac{\alpha+\beta}{2}\right)^+ = \begin{cases} \frac{\beta-\alpha}{2}e^{\sigma Y_T} + \alpha, & Y_T \geq \frac{1}{\sigma}\log\left(-\frac{2\alpha}{\beta-\alpha}\right), \\ 0, & Y_T \leq \frac{1}{\sigma}\log\left(-\frac{2\alpha}{\beta-\alpha}\right). \end{cases}$$

To represent the expected value of these random variables, for $\tau > 0$, $y \geq 0$, $w \geq 0$, we define the functions

$$\begin{aligned} g(\tau, y, w) &\triangleq \int_w^\infty \left(\frac{\beta-\alpha}{2}e^{\sigma x} + \alpha\right) \widetilde{\mathbb{P}}(Y_{t+\tau} \in dx | Y_t = y) \\ &= \frac{\beta-\alpha}{2}I_1 + \frac{\sigma(\beta-\alpha)}{4}I_2 + \alpha I_3 + \frac{\alpha\sigma}{2}I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi\tau}}\int_w^\infty e^{\sigma x - \frac{1}{2\tau}\left(y-x-\frac{1}{2}\sigma\tau\right)^2} dx \\ &= e^{\sigma y} N\left(\frac{y-w+\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right), \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{1}{\sqrt{2\pi\tau}}\int_w^\infty \int_{y+x}^\infty e^{-\frac{1}{2\tau}\left(v-\frac{1}{2}\sigma\tau\right)^2} dv dx \\ &= \sqrt{\frac{\tau}{2\pi}}\, e^{-\frac{1}{2\tau}\left(y+w-\frac{1}{2}\sigma\tau\right)^2} - \left(y+w-\frac{1}{2}\sigma\tau\right)\left[1 - N\left(\frac{y+w-\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right)\right], \\ I_3 &= \frac{1}{\sqrt{2\pi\tau}}\int_w^\infty e^{-\frac{1}{2\tau}\left(y-x-\frac{1}{2}\sigma\tau\right)^2} dx \;=\; N\left(\frac{y-w-\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right), \\ I_4 &= \frac{1}{\sqrt{2\pi\tau}}\int_w^\infty \int_{y+x}^\infty e^{-\sigma x - \frac{1}{2\tau}\left(v-\frac{1}{2}\sigma\tau\right)^2} dv dx \\ &= \frac{1}{\sigma}e^{-\sigma w}\left[1 - N\left(\frac{y+w-\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right)\right] - \frac{1}{\sigma}e^{-\sigma y}\left[1 - N\left(\frac{y+w+\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right)\right], \end{aligned}$$

and

$$\begin{aligned} h(\tau, y, w) &= \int_{-w}^0 \left(\beta - \frac{\beta-\alpha}{2}e^{-\sigma x}\right) \widetilde{\mathbb{P}}(Y_{t+\tau} \in dx | Y_t \in y) \\ &= \beta J_1 + \frac{\beta\sigma}{2}J_2 - \frac{\beta-\alpha}{2}J_3 - \frac{(\beta-\alpha)\sigma}{4}J_4, \end{aligned}$$

where

$$\begin{aligned}
J_1 &= \frac{1}{\sqrt{2\pi\tau}} \int_{-w}^{0} e^{\sigma y - \frac{1}{2\tau}\left(y-x+\frac{1}{2}\sigma\tau\right)^2} dx \\
&= e^{\sigma y}\left[N\left(\frac{y+w+\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right) - N\left(\frac{y+\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right)\right], \\
J_2 &= \frac{1}{\sqrt{2\pi\tau}} \int_{-w}^{0} \int_{y-x}^{\infty} e^{\sigma x - \frac{1}{2\tau}\left(v-\frac{1}{2}\sigma\tau\right)^2} dv dx \\
&= \frac{1}{\sigma}\left(1-e^{-\sigma w}\right) + \frac{1}{\sigma} e^{\sigma y} N\left(\frac{y+\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right) - \frac{1}{\sigma} N\left(\frac{y-\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right) \\
&\quad + e^{-\sigma w} N\left(\frac{y+w-\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right) - e^{\sigma y} N\left(\frac{y+w+\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right), \\
J_3 &= \frac{1}{\sqrt{2\pi\tau}} \int_{-w}^{0} e^{\sigma(y-x) - \frac{1}{2\tau}\left(y-x+\frac{1}{2}\sigma\tau\right)^2} dx \\
&= N\left(\frac{y+w-\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right) - N\left(\frac{y-\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right),
\end{aligned}$$

$$\begin{aligned}
J_4 &= \frac{1}{\sqrt{2\pi\tau}} \int_{-w}^{0} \int_{y-x}^{\infty} e^{-\frac{1}{2\tau}\left(v-\frac{1}{2}\sigma\tau\right)^2} dv dx \\
&= w + \sqrt{\frac{\tau}{2\pi}}\left[e^{-\frac{1}{2\tau}\left(y-\frac{1}{2}\sigma\tau\right)^2} - e^{-\frac{1}{2\tau}\left(y+w-\frac{1}{2}\sigma\tau\right)^2}\right] \\
&\quad + \left(y-\frac{1}{2}\sigma\tau\right) N\left(\frac{y-\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right) - \left(y+w-\frac{1}{2}\sigma\tau\right) N\left(\frac{y+w-\frac{1}{2}\sigma\tau}{\sqrt{\tau}}\right).
\end{aligned}$$

Consider the case $0 \le \alpha \le \beta$ or $\alpha \le 0 \le -\alpha \le \beta$, and take $x \ge \frac{1}{2}(\alpha+\beta)s > 0$. Define

$$y = \frac{1}{\sigma} \log\left[\frac{2}{\beta-\alpha}\left(\frac{x}{s}-\alpha\right)\right] \ge 0. \tag{6.5}$$

Then

$$\widetilde{\mathbb{E}}\left[\left(Z_T + \frac{\alpha+\beta}{2}\right)^+ \middle| Y_t = y\right] = g(T-t, y, 0) + h(t, y, w), \tag{6.6}$$

where

$$w = \frac{1}{\sigma} \log\left(\frac{2\beta}{\beta-\alpha}\right) \ge 0.$$

We set

$$d_\pm = \frac{y + w \pm \frac{1}{2}\sigma(T-t)}{\sqrt{T-t}} = \frac{1}{\overline{\sigma}} \log\left(\frac{2}{\beta-\alpha}\left(\frac{x}{s}-\alpha\right)\right) + \frac{1}{\overline{\sigma}} \log\left(\frac{2\beta}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2},$$

and evaluate (6.6) to obtain (6.1).

Consider next the case $\alpha < \beta \leq 0$ or $\alpha \leq 0 \leq \beta \leq -\alpha$, and again take $x \geq \frac{1}{2}(\alpha+\beta)s$. Although x may be negative, y defined by (6.5) is still nonnegative. We have

$$\widetilde{\mathbb{E}}\left[\left(Z_T + \frac{\alpha+\beta}{2}\right)^+ \middle| Y_t = y\right] = g(T-t, y, w), \tag{6.7}$$

where

$$w = \frac{1}{\sigma}\log\left(-\frac{2\alpha}{\beta-\alpha}\right) \geq 0.$$

We set

$$d_{\pm\pm} = \frac{y \pm w \pm \frac{1}{2}\sigma(T-t)}{\sqrt{T-t}} = \frac{1}{\overline{\sigma}}\log\left(\frac{2}{\beta-\alpha}\left(\frac{x}{s}-\alpha\right)\right) \pm \frac{1}{\overline{\sigma}}\log\left(-\frac{2\alpha}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2},$$

and evaluate (6.7) to obtain (6.3).

We next consider the case $0 \leq \alpha < \beta$ or $\alpha \leq 0 \leq -\alpha \leq \beta$ and $x \leq \frac{1}{2}(\alpha+\beta)s$. Define $\alpha' = -\beta$, $\beta' = -\alpha$, $x' = -x$, so $\alpha' < \beta' \leq 0$ or $\alpha' \leq 0 \leq \beta' \leq -\alpha'$ and $x' \geq \frac{1}{2}(\alpha'+\beta')s$. From Theorem 5.1 (generalized put-call parity), we have

$$V^{[\alpha,\beta]}(t,s,x) = x + V^{[\alpha',\beta']}(t,s,x').$$

Using (6.3) to evaluate $V^{[\alpha',\beta']}(t,s,x')$, we obtain (6.3). In a similar way, (6.4) is obtained from (6.1). $\diamond$

7 Vacation Options

We single out for special attention the *vacation call*, corresponding to $\alpha = 0$, $\beta = 1$, and the *vacation put*, coresponding to $\alpha = -1$, $\beta = 0$. For the passport option, one switches between short and long positions. If the underlying asset is a currency, this can be viewed as travel between different countries, which requires a *passport*. For the vacation call, one switches between a long and a zero position, which we view as a decision of whether or not to take a *vacation*.

Corollary 7.1 *For the vacation call, the optimal strategy is*

$$q_t^{opt} = I_{\{X_t^{q^{opt}} < \frac{1}{2}S_t\}}.$$

The value of the option is given by

$$\begin{aligned} VC(t,s,x) &= V^{[0,1]}(t,s,x) \\ &= \frac{1}{4}(1-\overline{\sigma}d_-)\,sN(-d_-) + \frac{1}{4\sqrt{2\pi}}\overline{\sigma}s\exp\left(-\tfrac{1}{2}d_-^2\right) + xN(d_+) \end{aligned}$$

for $x \geq \frac{1}{2}s$, where $d_\pm = \frac{1}{\overline{\sigma}}\log\left(\frac{4x}{s}\right) \pm \frac{\overline{\sigma}}{2}$, and

$$\begin{aligned} VC(t,s,x) &= V^{[0,1]}(t,s,x) \\ &= \frac{1}{4\sqrt{2\pi}}\overline{\sigma}s\exp\left(-\tfrac{1}{2}e_{+-}^2\right) - (x-s)N(-e_{++}) \\ &\quad -\frac{1}{4}(\overline{\sigma}e_{+-}+1)sN(-e_{+-}) + (x-s)N(-e_{-+}) + sN(-e_{--}) \end{aligned}$$

for $x \le \frac{1}{2}s$*, where* $e_{\pm\pm} = \frac{1}{\overline{\sigma}}\log\left(2 - \frac{2x}{s}\right) \pm \frac{1}{\overline{\sigma}}\log 2 \pm \frac{\overline{\sigma}}{2}$.

Remark 7.2 If $X_0 < \frac{1}{2}S_0$, the optimal strategy for the holder of the vacation call is to hold one share of stock until $X_t^{q^{opt}} = \frac{1}{2}S_t$. Until this occurs, (2.1) with $q \equiv 1$ implies

$$X_t^{q^{opt}} = S_t - e^{rt}(S_0 - X_0),$$

and so the switch to $q_t^{opt} = 0$ occurs only if $\frac{1}{2}S_t = e^{rt}(S_0 - X_0)$ sometime during the life of the option. This is equivalent to

$$W_t - \frac{1}{2}\sigma t = b \triangleq \frac{1}{\sigma}\log\left(2 - \frac{2X_0}{S_0}\right)$$

for some $t \in (0, T]$. Define the first passage time

$$\tau = \inf\left\{t \ge 0; W_t - \frac{1}{2}\sigma t = b\right\}.$$

The risk-neutral probability of a switch is (Karatzas & Shreve [8], (5.12) on page 197)

$$\mathbb{P}(\tau \le T) = \int_0^T \frac{b}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}\left(b + \frac{1}{2}\sigma t\right)^2} dt.$$

It is easily verified that

$$\frac{d}{dt}\left[N\left(-\frac{b + \frac{1}{2}\sigma t}{\sqrt{t}}\right) + e^{-\sigma b}N\left(-\frac{b - \frac{1}{2}\sigma t}{\sqrt{t}}\right)\right] = \frac{b}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}\left(b + \frac{1}{2}\sigma t\right)^2},$$

and hence the risk-neutral probability of a switch is

$$\mathbb{P}(\tau \le T) = N\left(-\frac{b + \frac{1}{2}\sigma T}{\sqrt{T}}\right) + e^{-\sigma b}N\left(-\frac{b - \frac{1}{2}\sigma T}{\sqrt{T}}\right).$$

Corollary 7.3 *For the vacation put, the optimal strategy is*

$$q_t^{opt} = -I_{\{X_t^{opt} \ge -\frac{1}{2}S_t\}}.$$

The value of the option is given by

$$\begin{aligned} VP(t,s,x) &= V^{[-1,0]}(t,s,x) \\ &= \frac{1}{4\sqrt{2\pi}}\overline{\sigma}s\exp\left(-\tfrac{1}{2}d_{+-}^2\right) + (x+s)\left[N(-d_{++}) + N(d_{-+})\right] \\ &\quad -\frac{1}{4}(\overline{\sigma}d_{+-} + 1)sN(-d_{+-}) - sN(d_{--}) \end{aligned}$$

for $x \ge -\frac{1}{2}s$*, where* $d_{\pm\pm} = \frac{1}{\overline{\sigma}}\log\left(\frac{2x}{s} + 2\right) \pm \frac{1}{\overline{\sigma}}\log 2 \pm \frac{\overline{\sigma}}{2}$*, and*

$$\begin{aligned} VP(t,s,x) &= V^{[-1,0]}(t,s,x) \\ &= \frac{1}{4\sqrt{2\pi}}\overline{\sigma}s\exp\left(-\tfrac{1}{2}e_{-}^2\right) + \frac{1}{4}(1 - \overline{\sigma}e_{-})sN(-e_{-}) + xN(-e_{+}), \end{aligned}$$

for $x \le \frac{1}{2}s$*, where* $e_{\pm} = \frac{1}{\overline{\sigma}}\log\left(-\frac{4x}{s}\right) \pm \frac{\overline{\sigma}}{2}$.

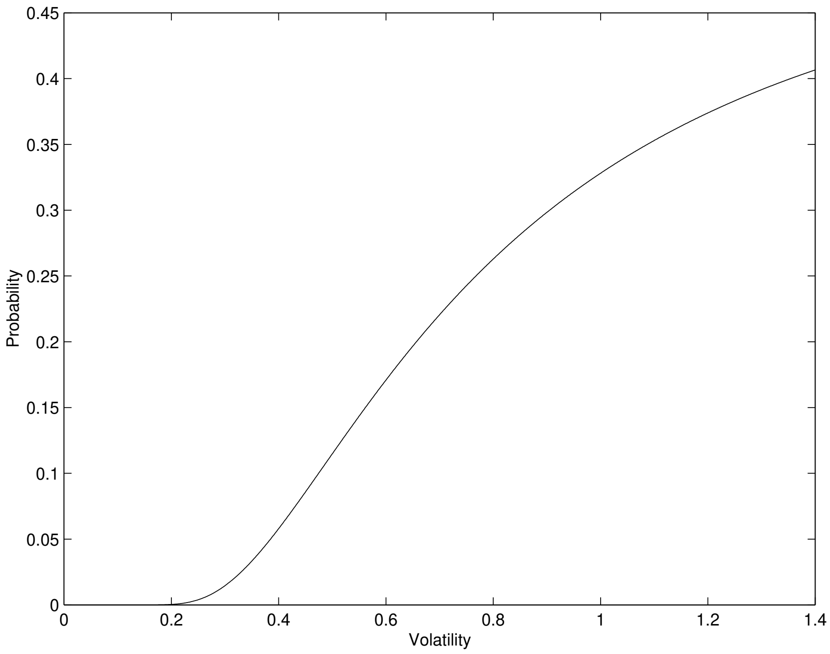

Figure 2: Probability of switching in the optimal strategy to a zero position in stock as a function of volatility when $X_0 = 0$ and $T = 1$.

8 The Seller's Hedge

The seller of an option on a traded account wants to ensure he can pay off the obligation implied by his short position, regardless of the strategy q used by the option holder. To do this, at each time t he must know the option holder's position q_t at that time, but does not need to know the holder's future strategy. In practice, the seller of the option manages the option holder's account. We shall see that the seller can exactly meet his obligation if the option holder uses the optimal policy q^{opt} given by (4.6), and if the option holder behaves suboptimally, then the option seller can consume wealth and still meet his obligation. We denote by p_t the number of shares of the underlying asset held by the seller at time t and by c_t the (nonnegative) rate at which the seller consumes at time t. Then the value of the seller's portfolio begins at

$$X_0^p = V^{[\alpha,\beta]}(0, S_0, X_0) = X_0 \tag{8.1}$$

and evolves according to the equation (cf. (2.1))

$$dX_t^p = rX_t^p dt + p_t \sigma S_t dW_t - c_t dt. \tag{8.2}$$

The hedging strategy must guarantee that $X_T^p \geq (X_T^q)^+$ almost surely.

Theorem 8.1 *With X_t^p, $0 \le t \le T$ determined by (8.1), (8.2), and with*

$$\begin{aligned} p_t &= V_s^{[\alpha,\beta]}(t,S_t,X_t^q) + q_t V_x^{[\alpha,\beta]}(t,S_t,X_t^q) \\ c_t &= \frac{1}{2}\sigma^2 S_t^2 \left(2(q_t^{opt} - q_t) V_{sx}^{[\alpha,\beta]} + \left((q_t^{opt})^2 - q_t^2 \right) V_{xx}^{[\alpha,\beta]} \right) \;\ge\; 0, \end{aligned}$$

we have $X_T^p = (X_T^q)^+$ almost surely.

Proof. We simplify notation by writing $V = V^{[\alpha,\beta]}$. In the following equation, the argument of V and its derivatives is always (t, S_t, X_t^q). Recall the HJB equation (2.3)

$$\begin{aligned} -rV + V_t + rS_tV_s + rX_t^qV_x + \frac{1}{2}\sigma^2 S_t^2 \max_{\alpha \le q \le \beta} \left(V_{ss} + 2qV_{sx} + q^2 V_{xx} \right) &= \\ -rV + V_t + rS_tV_s + rX_t^qV_x + \frac{1}{2}\sigma^2 S_t^2 \left(V_{ss} + 2q_t^{opt}V_{sx} + (q_t^{opt})^2 V_{xx} \right) &= 0. \end{aligned}$$

From Itô's rule and this equation, we have

$$\begin{aligned} & d\left(e^{-rt}V(t,S_t,X_t^q)\right) \\ &= e^{-rt}\sigma S_t \left[V_s + q_tV_x\right] dW_t \\ &\quad + e^{-rt}\left[-rV + V_t + rS_tV_s + rX_t^qV_x + \frac{1}{2}\sigma^2 S_t^2 \left(V_{ss} + 2q_tV_{sx} + q_t^2 V_{xx} \right) \right] dt \\ &= e^{-rt} p_t \sigma S_t dW_t - e^{-rt} c_t dt \\ &= d\left(e^{-rt} dX_t^p\right). \end{aligned}$$

Integrating both sides from $t = 0$ to $t = T$, we obtain

$$e^{-rT}\left(X_T^q\right)^+ = e^{-rT}V(T, S_T, X_T^q) = e^{-rT}X_T^p,$$

and the theorem is proved. ◇

9 Conclusion

We have introduced a family of options on traded accounts, which includes passport options. We determine the optimal strategy for the buyer of these options. We also show how the seller can hedge his short position, and may gain from suboptimal behavior on the part of the buyer. We highlighted two options, the vacation call and the vacation put. These are similar in price to the American call and the American put, respectively, but have more optionality.

References

[1] Andersen, L., Andreasen, J., Brotherton-Ratcliffe, R., "The passport option", *The Journal of Computational Finance* Vol. 1, No. 3, Spring 1998, 15–36.

[2] DELBAEN, F. & YOR, M. "Passport options", preliminary manuscript, Dept. Math., ETH-Zürich, http://www.math.ethz.ch/~delbaen/.

[3] ENGELBERT, H. J. & SCHMIDT, W. "On solutions of stochastic differential equations without drift", *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 68, 287–314, 1985.

[4] FLEMING, W., RISHEL, R., *Deterministic and Stochastic Optimal Control,* Springer Verlag, 1975.

[5] HAJEK, B., "Mean stochastic comparison of diffusions", *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* 68, 315–329, 1985.

[6] HENDERSON, V., HOBSON, D., "Local time, coupling and the passport option", *Finance and Stochastics*, Vol. 4, No. 1, 1999, to appear.

[7] HYER, T., LIPTON-LIFSCHITZ, A., PUGACHEVSKY, D., "Passport to success", *Risk*, Vol. 10, No. 9, September 1997, 127 – 131.

[8] KARATZAS, I., SHREVE, S., *Brownian Motion and Stochastic Calculus,* Springer Verlag, Second Edition, 1996.

[9] NAGAYAMA, I., "Pricing of passport option", J. Math. Sci. Univ. Tokyo 5, 747–785, 1999.