Probability Theory II (G6106)
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http://stat.columbia.edu/∼porbanz/G6106S16.html

Peter Orbanz
porbanz@stat.columbia.edu
Florian Stebegg
florian@stat.columbia.edu

Homework 9

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Problem 1

Prove Lemma 4.8 (properties of outer measure) in the class notes.

Problem 2

Recall that the gamma distribution with parameters (α, λ) is the distribution on $(0, \infty)$ with Lebesgue density $p(x) = \Gamma(\lambda)^{-1}\alpha^\lambda x^{\lambda-1}e^{-\alpha x}$. Suppose X and Y are independent gamma variables with parameters (α, λ_y) and (α, λ_x). Show that

$$\tfrac{X}{Y} \perp\!\!\!\perp X+Y \qquad \text{and} \qquad \tfrac{X}{X+Y} \perp\!\!\!\perp X+Y\ .$$

Problem 3

Many stochastic processes can be interpreted as distributions on functions, i.e. as random variables whose sample space is a space of functions. Suppose we are interested in functions $f : \mathbb{R}_+ \to \mathbb{R}$. The set of all such functions is the product space $\mathbb{R}^{\mathbb{R}_+}$. More generally, the set of all functions from $\mathbb{T}$ to $\mathbf{X}$ is the product set $\mathbf{X}^{\mathbb{T}}$. One of the main technical obstacles in the construction of stochastic processes is that $\mathbb{T}$ is often uncountable (such as in the case $\mathbb{T} = \mathbb{R}_+$ above); if so, the product set $\mathbf{X}^{\mathbb{T}}$ is still well-defined, but the product σ-algebra and product topology are becoming too coarse. The purpose of this problem, and of problem 4 below, is to understand this phenomenon better.

Let $\mathbf{X}$ be a second-countable Hausdorff space with Borel σ-algebra $\mathcal{B}(\mathbf{X})$. Let $\mathbb{T}$ be an arbitrary set, and consider the product space

$$\mathbf{X}^{\mathbb{T}} := \prod_{t\in\mathbb{T}} \mathbf{X}_t\ ,$$

in which each factor is a copy of $\mathbf{X}_t := \mathbf{X}$ of $\mathbf{X}$. For each t, denote by $\mathrm{pr}_t : \mathbf{X}^{\mathbb{T}} \to \mathbf{X}_t$ the projection map (cf. Definition 2.4 in the class notes). We write $\mathcal{B}(\mathbf{X})^{\mathbb{T}}$ for the product σ-algebra (the smallest σ-algebra on $\mathbf{X}^{\mathbb{T}}$ which makes all projections measurable), and $\mathcal{B}(\mathbf{X}^{\mathbb{T}})$ for the Borel σ-algebra generated by the product topology.

Question (a): Show that $\mathcal{B}(\mathbf{X})^{\mathbb{T}} \subset \mathcal{B}(\mathbf{X}^{\mathbb{T}})$.

Question (b): If $\mathbb{T}$ is countable, show that equality holds, i.e. $\mathcal{B}(\mathbf{X})^{\mathbb{T}} = \mathcal{B}(\mathbf{X}^{\mathbb{T}})$.

Question (c): Show that $\mathcal{B}(\mathbf{X})^{\mathbb{T}} \neq \mathcal{B}(\mathbf{X}^{\mathbb{T}})$ can hold if $\mathbb{T}$ is uncountable. In particular, show that, if $\mathbf{X}$ contains more than one element, the singleton set $\{x\}$, for $x \in \mathbf{X}^{\mathbb{T}}$, are not in $\mathcal{B}(\mathbf{X})^{\mathbb{T}}$.

Problem 4

Let $\mathbb{T}$ be an uncountable set. Let Ω be a set, and $\mathcal{A}_t$ a σ-algebra on Ω for each $t \in \mathbb{T}$.

Question (a): Show that, for every $A \in \sigma\Big(\bigcup_{t\in\mathbb{T}}\mathcal{A}_t\Big)$, there exists a countable subset $I \in \mathbb{T}$ such that $A \in \sigma\Big(\bigcup_{t\in I}\mathcal{A}_t\Big)$.

Question (b): Let $\mathbf{X}^{\mathbb{T}}$ again be the product space in Problem 3. Show that an event A is measurable in the product σ-algebra $\mathcal{B}(\mathbf{X})^{\mathbb{T}}$ if and only if it depends only on a countable number of coordinates. That is: Show that, whenever $A \in \mathcal{B}(\mathbf{X})^{\mathbb{T}}$, there exists a countable subset $I \subset \mathbb{T}$ and sets $A_t \in \mathcal{B}(\mathbf{X}_t)$, for $t \in I$, such that $x \in A$ iff $\mathrm{pr}_t x \in A_t$ for all $t \in A$.