Probability Theory II (G6106)
Spring 2016
http://stat.columbia.edu/∼porbanz/G6106S16.html

Peter Orbanz
porbanz@stat.columbia.edu
Florian Stebegg
florian@stat.columbia.edu

Homework 6

Due: 30 March 2016

Problem 1 (Conditional probabilities define measures)

Let Y be a random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with values in a measurable space $(\mathcal{Y}, \mathcal{A}_Y)$. Recall that we define the conditional probability of a given set $A \in \mathcal{A}$ as

$$\mathbb{P}(A|Y = y) := \mathbb{E}[\mathbb{I}_A|Y = y] . \tag{1}$$

Question (a): Show that, for any $A \in \mathcal{A}$ and $C \in \mathcal{A}_Y$,

$$\mathbb{P}(A \cap \{Y \in C\}) = \int_C \mathbb{P}(A|Y = y)P_Y(dy) , \tag{2}$$

where P_Y denotes the law of Y.

Question (b): Show that, for any fixed value $y \in \mathcal{Y}$, the function $A \mapsto \mathbb{P}(A|Y = y)$ is P_Y-almost surely a probability measure on $(\Omega, \mathcal{A})$.

Problem 2 (Compact classes)

Let $\mathbf{X}$ be a Hausdorff space. Let $\mathcal{K}$ be the set of all compact sets in $\mathbf{X}$.

Question: Show $\mathcal{K}$ is a compact class.

Problem 3 (Independence)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$ two sub-σ-algebras. We again use the definition

$$\mathbb{P}(A|\mathcal{C})(\omega) := \mathbb{E}[\mathbb{I}_A|\mathcal{C}](\omega) \tag{3}$$

for the conditional probability of A given $\mathcal{C}$. Show that the σ-algebras $\mathcal{B}$ and $\mathcal{C}$ are independent if and only if

$$\forall B \in \mathcal{B} : \qquad \mathbb{P}(B|\mathcal{C}) = \mathbb{P}(B) \qquad \text{almost surely} . \tag{4}$$

Note: Recall the definition of independent σ-algebras from Probability I [e.g. Jacod & Protter, Chapter 10].

Problem 4 (Conditional densities)

Let X and Y be random variables with values $\mathbb{R}$, with joint law P, and let λ denote Lebesgue measure on $\mathbb{R}$. Let $\mathbf{p}$ be a version of the conditional distribution of X given Y, that is, $\mathbf{p}(A, y) = \mathbb{P}(X \in A|Y = y)$ almost surely. Suppose $f(x, y)$ is a density of the joint distribution P with respect to $\lambda \otimes \lambda$, and $f(y) := \int_{\mathbb{R}} f(x, y)\lambda(dx)$.

Question: Show that, if $f(y) > 0$ for all $y \in \mathbb{R}$,

$$f(x|y) := \frac{f(x, y)}{f(y)} \tag{5}$$

is a density of $\mathbf{p}(\bullet, y)$ with respect to λ for all y.