## Probability Theory II (G6106)

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http://stat.columbia.edu/∼porbanz/G6106S16.html

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# Homework 2

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### Problem 1

Let $(\mathbb{T}, \preceq)$ be a directed set and $\mathcal{F} = (\mathcal{F}_s)_{s\in\mathbb{T}}$ a filtration. For each $i = 1, \ldots, n$, let $(X^i_s, \mathcal{F}_s)_{s\in\mathbb{T}}$ be a martingale.

**Question:** Show that $(\max_{i\leq n} X^i_s, \mathcal{F}_s)$ is a submartingale.

### Problem 2 (Azuma's inequality)

Let $(X_t, \sigma(X_t))_{t\in\mathbb{N}}$ be a martingale, $(c_t)_{t\in\mathbb{N}}$ be a sequence of non-negative constants, and define $\mu := \mathbb{E}[X_t]$. (Note $\mu$ does not depend on $t$.) The purpose of this problem is to prove Azuma's inequality, recall: If

$$|X_{t+1} - X_t| \leq c_{t+1} \quad \text{for all } t \text{ and} \quad |X_1 - \mu| \leq c_1 \ , \tag{1}$$

then

$$\mathbb{P}\{|X_t - \mu| \geq \lambda\} \leq 2\exp\Big(-\frac{\lambda^2}{2\sum_{s=1}^{t} c_s^2}\Big) \qquad \text{for all } \lambda > 0 \ . \tag{2}$$

We will use the following general version of Markov's inequality: For any real-valued random variable $X$ and any monotonically increasing function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$,

$$\mathbb{P}[|X| \geq \lambda] \leq \frac{\mathbb{E}[f(|X|)]}{f(\lambda)} \qquad \text{for all } \lambda > 0 \text{ with } f(\lambda) > 0 \ . \tag{3}$$

To show the inequality holds, first consider a random variable $Y$ with values in $[-1, 1]$, and show the following:

**Question (a):** There is a random variable $Z$ with values in $\{-1, 1\}$ such that $\mathbb{E}[Y|Z] = Z$.

**Question (b):** If additionally $\mathbb{E}[Y] = 0$, then $\mathbb{E}[\exp(\lambda Y)] \leq \cosh(\lambda) \leq \exp(\lambda^2/2)$.

Next, consider the martingale $(X_t)$ and assume the hypothesis (1) holds.

**Question (c):** Show that

$$\mathbb{E}[e^{\lambda X_t}] \leq \exp\Big(\frac{1}{2}\lambda^2 \sum_{s=1}^{t} c_s^2\Big) \ . \tag{4}$$

**Question (d):** Deduce (2).

**Hint**: Use Jensen's inequality in (b). To apply the Markov inequality, use $f(a) = \exp(ab)$ for a suitable $b$.

### Problem 3 (Potentials)

Let $(X_t, \mathcal{F}_t)$ be a positive, discrete-time supermartingale. (Such a process is sometimes called a *potential*.)

**Question:** Show that $\lim_t \mathbb{E}[X_t] = 0$ implies $X_t \to 0$ almost surely and in $\mathbf{L}_1$.