Probability Theory II (G6106)
Spring 2016
http://stat.columbia.edu/∼porbanz/G6106S16.html

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Homework 1

Due: 3 February 2016

Homework submission: Please leave your homework in Florian's mailbox **no later than start of class** (10:10am) on the due date.

Problem 1 (Directed sets)

Question (a): Let $\mathcal{X}$ be a set and $x \in \mathcal{X}$. Recall that a **neighborhood** of x is any set $A \subset \mathcal{X}$ which contains x. Let $\mathbb{T}$ be the set of all neighborhoods of a fixed point x, ordered by reverse inclusion, i.e. $A \preceq B$ iff $A \supset B$. Show that $(\mathbb{T}, \preceq)$ is a directed set.

Question (b): Let $(\mathbb{T}_1, \preceq_1)$ and $(\mathbb{T}_1, \preceq_1)$ be directed sets. Show that the Cartesian product $\mathbb{T}_1 \times \mathbb{T}_2$ is directed in the partial order defined by $(s_1, s_2) \preceq (t_1, t_2)$ iff $s_1 \preceq_1 t_1$ and $s_2 \preceq_2 t_2$.

Problem 2 (A martingale indexed by partitions)

Let $(\Omega, \mathcal{A})$ be a measurable space. A **finite measurable partition**finite measurable partition $s = (A_1, \ldots, A_n)$ of Ω is a subdivision of Ω into a finite number of disjoint measurable sets A_i whose union is Ω. We say that a partition $t = (B_1, \ldots, B_m)$ is a **refinement**refinement of another partition $s = (A_1, \ldots, A_n)$ if every set B_j in t is a subset of some set A_i in s; in words, t can be obtaine from s by splitting sets in s further, without changing any of the existing set boundaries in s.

Let $\mathbb{T}$ be the set of all finite measurable partitions of Ω, and defined as binary relation $\preceq$ as

$$s \preceq t \qquad \Leftrightarrow \qquad t \text{ is a refinement of } s\ .$$

Question (a): Show that $\preceq$ is a partial order on $\mathbb{T}$.

Question (b): Show that the partially ordered set $(\mathbb{T}, \preceq)$ is directed.

Later on in the lecture, we will use this construction to prove the Radon-Nikodym theorem on the existence of densities. We anticipate a part of the proof in this problem (you can find the proof in Chapter 1.9 of the class notes, but you are *not* required to read ahead to solve this problem). The proof idea is to "discretize" the density f of a measure μ with respect to a probability measure P on finite partitions s as above. To this end, let $s \in \mathbb{T}$, so s is of the form $s = (A_1, \ldots, A_n)$ for some $n \geq 2$. Define a finite σ-algebra

$$\mathcal{F}_s := \sigma(s) = \sigma(A_1, \ldots, A_n)\ .$$

Now let μ be a finite measure and P a probability measure, both defined $(\Omega, \mathcal{A})$. For each s, we define the function

$$Y_s(x) := \sum_{j=1}^{n} f_s(A_j)\mathbb{I}_{A_j}(x) \qquad \text{where } f_s(A_j) := \begin{cases} \frac{\mu(A_j)}{P(A_j)} & P(A_j) > 0 \\ 0 & P(A_j) = 0 \end{cases}\ .$$

Note that Y_s is a real-valued, measurable function defined on a probability space $(\Omega, \mathcal{A}, P)$, and hence a real-valued random variable (even though it may not seem particularly random).

Question (c): Show that $(Y_s, \mathcal{F}_s)_{s \in \mathbb{T}}$ is a martingale.

Problem 3 Stopping times and filtrations

Prove Lemma 1.7. That is, let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration, and S, T stopping times.

Question (a): Show $\mathcal{F}_T$ is a σ-algebra.

Question (b): Suppose $S \leq T$ almost surely. Show that $\mathcal{F}_S \subset \mathcal{F}_T$.

Question (c): Let $(X_n)_{n\in\mathbb{N}}$ be a random sequence adapted to $\mathcal{F}$, where each X_n takes values in a measurable space $(\mathbf{X}, \mathcal{C})$, and assume $T < \infty$ almost surely. Show that X_T is $\mathcal{F}_T$-measurable.

Problem 4 Convex images of (sub)martingales

Let $X = (X_t, \mathcal{F}_t)_{t\in\mathbb{N}}$ be an adapted process, and $f : \mathbb{R} \to \mathbb{R}$ a convex function, with $f(X_t)$ integrable for all t.

Question (a): Item $(f(X_t), \mathcal{F}_t)$ is a submartingale if X is a martingale.

Question (b): Item $(f(X_t), \mathcal{F}_t)$ is a submartingale if X is a submartingale and f non-decreasing.

Hint: Use Jensen's inequality for conditional expectations (recall: $f(\mathbb{E}[X|\mathcal{C}]) \leq_{\text{a.s.}} \mathbb{E}[f(X)|\mathcal{C}]$ for any convex f).